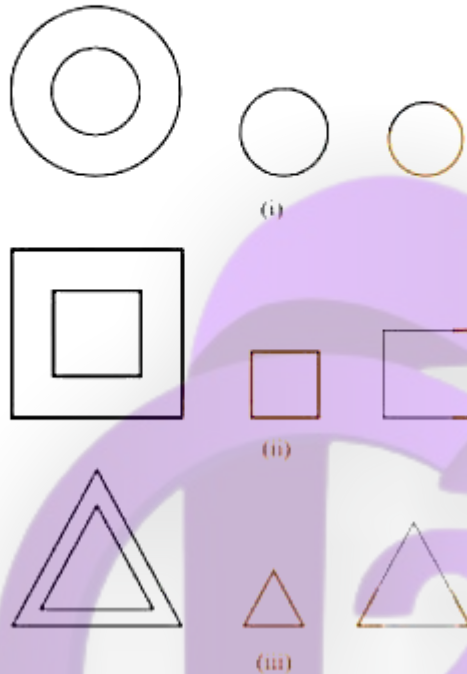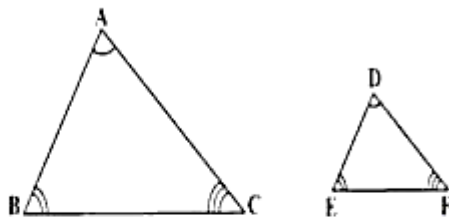## Triangles

Similar Figures

The above are similar figures.

- All congruent figures are similar but similar figures need not be congruent.
- Two regular polygons of the same number of sides are similar, if
  (i) their corresponding angles are equal and
  (ii) their corresponding sides are in the same ratio (or proportion).

Similarity of Triangles:

- Two triangles are similar, if
  a) their corresponding angles are equal and
  b) their corresponding sides are in the same ratio (or proportion).

- Criterion of similarity:
  In $\triangle ABC$ and $\triangle DEF$, if
  (i) if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and
  (ii) $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$, then the two triangles are similar.

Theorems in similarity (SSS, AA, SAS, BPT)

AA criterion of similarity or AAA criterion:

If in two triangles, the corresponding angles are equal, then their corresponding sides are in the same ratio (or proportion) and hence the two triangles are similar.

This criterion is referred to as the AAA (Angle-Angle-Angle) criterion of similarity of two triangles.

This theorem can be proved by taking two triangles ABC and DEF such that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

Cut DP = AB and DQ = AC and join PQ.

So, $\triangle ABC \cong \triangle DPQ$

This gives $\angle B = \angle P = \angle E$ and PQ $\parallel$ EF

Therefore, $\dfrac{DP}{PE} = \dfrac{DQ}{QF}$

i.e. $\dfrac{AB}{DE} = \dfrac{AC}{DF}$

Similarly, $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ and so

$$\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$$

SSS criterion of similarity:

If in two triangles, the sides of one triangle are proportional to (i.e. in the same ratio of) the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similar.

This criterion is referred to as the SSS (Side-Side-Side) similarity criterion for two triangles.

This theorem can be proved by taking two triangles ABC and DEF such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}\,(< 1)$$
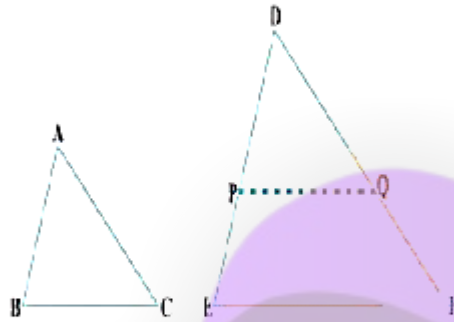
Cut DP = AB and DQ = AC and join PQ.

It can be seen that $\dfrac{DP}{PE} = \dfrac{DQ}{QF}$ and

PQ || EF

So, ∠P = ∠E and ∠Q = ∠F

Therefore, $\dfrac{DP}{DE} = \dfrac{DQ}{DF} = \dfrac{PQ}{EF}$

So, $\dfrac{DP}{DE} = \dfrac{DQ}{DF} = \dfrac{BC}{EF}$

So, BC = PQ

SAS criterion of similarity:
If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.

This criterion is referred to as the SAS (Side-Angle-Side) similarity criterion for two triangles.

As before, this theorem can be proved by taking two triangles ABC and DEF such that

$$\frac{AB}{DE} = \frac{AC}{DF}\,(< 1)\text{ and }\angle A = \angle D.$$

Cut DP = AB, DQ = AC and join PQ.
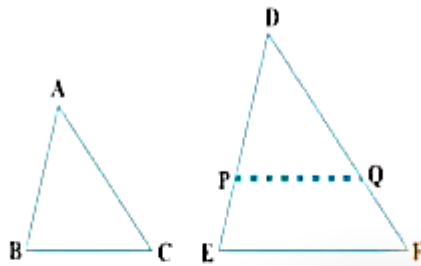
Now, PQ $\parallel$ EF and $\triangle ABC \cong \triangle DPQ$

So, $\angle A = \angle D, \angle B = \angle P$ and $\angle C = \angle Q$

Therefore, $\triangle ABC \sim \triangle DEF$

Basic Proportionality theorem:
If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.

Proof: We are given a triangle ABC in which a line parallel to side BC intersects the other two sides AB and AC at D and E respectively.
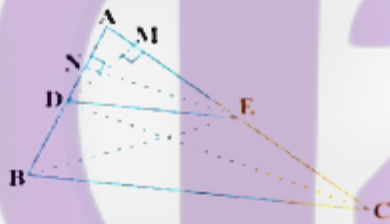
We need to prove that $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Let us join BE and CD and then draw
DM $\perp$ AC and EN $\perp$ AB

Now, area of $\triangle ADE = \dfrac{1}{2} \times$ base $\times$ height

$= \dfrac{1}{2} \times AD \times EN$

Recall from Class IX, that area of $\triangle ADE$ is denoted as ar(ADE).

So, ar(ADE) $= \dfrac{1}{2} \times AD \times EN$

Similarly, ar(BDE) $= \dfrac{1}{2} \times DB \times EN$

ar(ADE) $= \dfrac{1}{2} \times AE \times DM$ and

$ar(DEC) = \dfrac{1}{2} \times EC \times DM$

Therefore, $\dfrac{ar(ADE)}{ar(BDE)} = \dfrac{\dfrac{1}{2} \times AD \times EN}{\dfrac{1}{2} \times DB \times EN}$

$= \dfrac{AD}{DB} \ldots (1)$

and $\dfrac{ar(ADE)}{ar(DEC)} = \dfrac{\dfrac{1}{2} \times AE \times DM}{\dfrac{1}{2} \times EC \times DM}$

$= \dfrac{AE}{EC} \ldots (2)$

Note that $\triangle BDE$ and $\triangle DEC$ are on the same base DE and between the same parallels BC and DE.
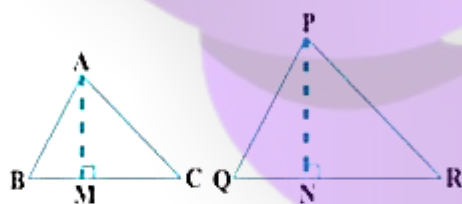So, ar(BDE) = ar(DEC)...(3)
Therefore, from (1), (2) and (3), we have:

$\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Areas of Similar Triangles:
The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Proof: We are given two triangles ABC and PQR such that $\triangle ABC \sim \triangle PQR$

We need to prove that

$\dfrac{ar(ABC)}{ar(PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{CA}{RP}\right)^2$

For finding the areas of the two triangles, we draw altitudes AM and PN of the triangles.

Now, ar(ABC) $= \dfrac{1}{2} \times BC \times AM$

and ar(PQR) $= \dfrac{1}{2} \times QR \times PN$

So, $\dfrac{ar(ABC)}{ar(PQR)} = \dfrac{\dfrac{1}{2} \times BC \times AM}{\dfrac{1}{2} \times QR \times PN}$ ... (1)

Now, in $\triangle ABM$ and $\triangle PQN$,

$\angle B = \angle Q$ (As $\triangle ABC \sim \triangle PQR$)

and $\angle M = \angle N$ (Each is 90°)

So, $\triangle ABM \sim \triangle PQN$ (AA similarity criterion)

Therefore, $\dfrac{AM}{PN} = \dfrac{AB}{PQ}$ ... (2)

Also, $\triangle ABC \sim \triangle PQR$ (Given)

So, $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{CA}{RP}$ ... (3)

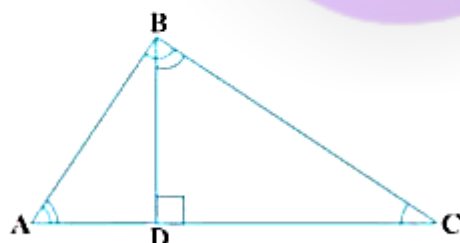Therefore, $\dfrac{ar(ABC)}{ar(PQR)} = \dfrac{AB}{PQ} \times \dfrac{AM}{PN}$

[From (1) & (3)]

$= \dfrac{AB}{PQ} \times \dfrac{AB}{PQ}$ [From (2)]

$= \left(\dfrac{AB}{PQ}\right)^2$

Now using (3) we get:

$\dfrac{ar(ABC)}{ar(PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{CA}{RP}\right)^2$

Proof of Pythagoras Theorem Using Similarity

You may note that in $\triangle ADB$ and $\triangle ABC$

$\angle A = \angle A$

and $\angle ADB = \angle ABC$

So, $\triangle ADB \sim \triangle ABC$

Similarly, $\triangle BDC \sim \triangle ABC$

From (1) and (2), triangles on both sides of the perpendicular BD are similar to the whole triangle ABC.

Also, since $\triangle ADB \sim \triangle ABC$

and $\triangle BDC \sim \triangle ABC$

So, $\triangle BDC \sim \triangle BDC$

If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole triangle and to each other.

Let us now apply this theorem in proving the Pythagoras Theorem:
In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Proof: We are given a right triangle ABC right angled at B.
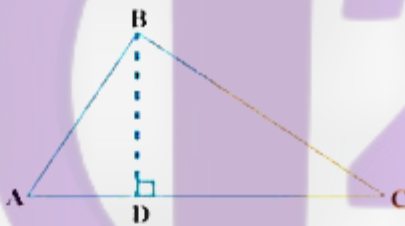
We need to prove that $AC^2 = AB^2 + BC^2$

Let us draw $BD \perp AC$.

Now, $\triangle ADB \sim \triangle ABC$

So, $\dfrac{AD}{AB} = \dfrac{AB}{AC}$ (Sides are proportional)

or $AD \cdot AC = AB^2$ ... (1)

Also, $\triangle BDC \sim \triangle ABC$

So, $\dfrac{CD}{BC} = \dfrac{BC}{AC}$

or $CD \cdot AC = BC^2$ ... (2)

Adding (1) and (2),

$AD \cdot AC + CD \cdot AC = AB^2 + BC^2$

Or $AC (AD + CD) = AB^2 + BC^2$

Or $AC \cdot AC = AB^2 + BC^2$

Or $AC^2 = AB^2 + BC^2$

Converse of Pythagoras Theorem:
In a right triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.

Proof: Here we are given a triangle ABC in which $AC^2 = AB^2 + BC^2$
We need to prove that $\angle B = 90°$
To start with, we construct a $\triangle PQR$ right angled at Q such that PQ = AB and QR = BC
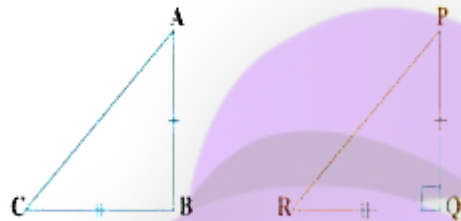
Now, from $\triangle PQR$, we have:
$PR^2 = PQ^2 + QR^2$ (Pythagoras Theorem, as $\angle Q = 90°$)
Or $PR^2 = AB^2 + BC^2$ (By construction). (1)
But $AC^2 = AB^2 + BC^2$ (Given)... (2)
So, AC = PR [From (1) and (2)] ... (3)
Now, in $\triangle ABC$ and $\triangle PQR$,
AB = PQ (By construction)
BC = QR (By construction)
AC = PR [Proved in (3) above]
So, $\triangle ABC \cong \triangle PQR$ (SSS congruence)
Therefore, $\angle B = \angle Q$ (CPCT)
But $\angle Q = 90°$ (By construction)
So, $\angle Q = 90°$