APPLICATION OF DERIVATIVES

1. DERIVATIVE AS RATE OF CHANGE

In various fields of applied mathematics one has the quest to know the rate at which one variable is changing, with respect to other. The rate of change naturally refers to time. But we can have rate of change with respect to other variables also.

An economist may want to study how the investment changes with respect to variations in interest rates.

A physician may want to know, how small changes in dosage can affect the body's response to a drug.

A physicist may want to know the rate of change of distance with respect to time.

All questions of the above type can be interpreted and represented using derivatives.

Definition :

The **average rate of change** of a function $f(x)$ with respect to x over an interval [a, a + h] is defined as $\frac{f(a+h)-f(a)}{h}$.

Definition :

The **instantaneous rate of change** of f with respect to x is defined as

$$f'(x)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$, provided the limit exists.

Note...

To use the word 'instantaneous', x may not be representing time. We usually use the word 'rate of change' to mean 'instantaneous rate of change'.

2. EQUATIONS OF TANGENT & NORMAL

(I) The value of the derivative at P (x_1, y_1) gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1)=\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \text{Slope of tangent at}$$

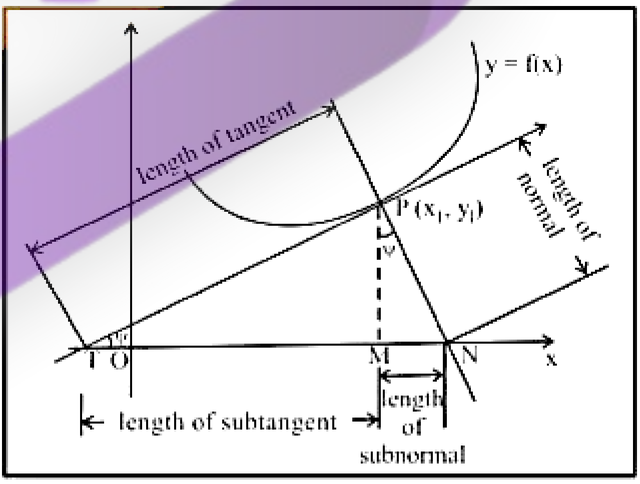
P (x_1, y_1) = m (say).

(II) Equation of tangent at (x_1, y_1) is ;

$$(y-y_1)=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}\times(x-x_1)$$

(III) Equation of normal at (x_1, y_1) is :

$$(y-y_1)=\left(\frac{-1}{\frac{dy}{dx}}\right)_{(x_1,y_1)}\times(x-x_1)$$



Note..

1. The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.
2. If the tangent at any point P on the curve is parallel to X-axis then dy/dx = 0 at the point P.
3. If the tangent at any point on the curve is parallel to Y-axis, then dy/dx = ∞ or dx/dy = 0.
4. If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ±1.
5. If the tangent at any point makes equal intercept on the coordinate axes then dy/dx = −1.
6. Tangent to a curve at the point P (x_1, y_1) can be drawn even though dy/dx at P does not exist. e.g. x = 0 is a tangent to $y = x^{2/3}$ at (0, 0).
7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$.

(IV) (a) Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(b) Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(d) Length of Subnormal (MN) = $y_1 f'(x_1)$

(V) Differential :

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tan x then $dy = \sec^2 x\, dx$.

In general $dy = f'(x)\, dx$.

Note..

$d(c) = 0$ where 'c' is a constant.

$d(u + v - w) = du + dv - dw$

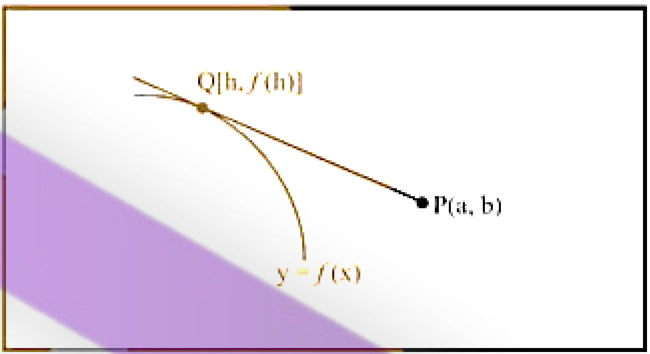
$d(uv) = udv + vdu$

* The relation $dy = f'(x)\, dx$ can be written as $\dfrac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

3. TANGENT FROM AN EXTERNAL POINT

Given a point P (a, b) which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve $y = f(x)$, passing through (a, b) can be found by solving for the point of contact Q.

And equation of tangent is $y - b = \dfrac{f(h) - b}{h - a}(x - a)$

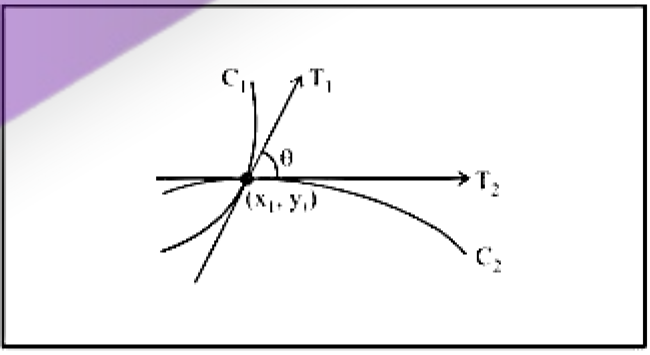


4. ANGLE BETWEEN THE CURVES

Angle between two intersecting curves is defined as the acute angle between their tangents or the normals at the point of intersection of two curves.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1).



Note..

(i) The angle is defined between two curves if the curves are intersecting. This can be ensured by finding their point of intersection or by graphically.

(ii) If the curves intersect at more than one point then angle between curves is found out with respect to the point of intersection.

(iii) Two curves are said to be **orthogonal** if angle between them at each point of intersection is right angle i.e. $m_1 m_2 = -1$.

5. SHORTEST DISTANCE BETWEEN TWO CURVES

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

6. ERRORS AND APPROXIMATIONS

(a) Errors

Let $y = f(x)$

From definition of derivative. $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

$\frac{\delta y}{\delta x} = \frac{dy}{dx}$ approximately

or $\delta y = \left(\frac{dy}{dx}\right) . \delta x$ approximately

Definition :

(i) δx is known as absolute error in x.

(ii) $\frac{\delta x}{x}$ is known as relative error in x.

(iii) $\frac{\delta x}{x} \times 100$ is known as percentage error in x.

Note..

δx and δy are known as differentials.

(b) Approximations

From definition of derivative,

$\therefore$ Derivative of $f(x)$ at $(x = a) = f'(a)$

or $f'(a) = \lim_{\delta x \to 0} \frac{f(a + \delta x) - f(a)}{\delta x}$

or $\frac{f(a + \delta x) - f(a)}{\delta x} \to f'(a)$ (approximately)

$f(a + \delta x) = f(a) + \delta x f'(a)$ (approximately)

7. DEFINITIONS

1. A function $f(x)$ is called an **Increasing Function** at a point x = a if in a sufficiently small neighbourhood around x = a we have

$f(a + h) > f(a)$

$f(a - h) < f(a)$

Similarly **Decreasing Function if**

$f(a + h) < f(a)$

$f(a - h) > f(a)$

Above statements hold true irrespective of whether f is non derivable or even discontinuous at x = a

2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points). A function decreasing in an interval (a, b) is similarly defined.

3. A function which in a given interval is increasing or decreasing is called **"Monotonic"** in that interval.

4. Tests for increasing and decreasing of a function at a point :

If the derivative $f'(x)$ is positive at a point x = a, then the function $f(x)$ at this point is increasing. If it is negative, then the function is decreasing.

Note...

Even if $f'(a)$ is not defined, f can still be increasing or decreasing. (Look at the cases below).

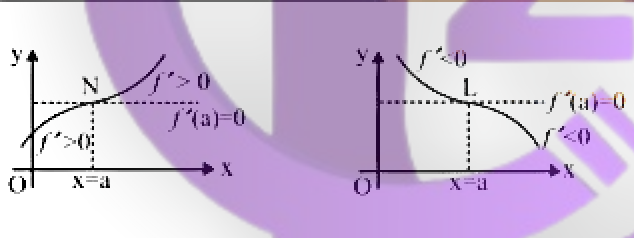
Note : $f'(c)$ is not defined in both the cases.

Note...

If $f'(a) = 0$, then for $x = a$ the function may be still increasing or it may be decreasing as shown. It has to be identified by a seperate rule.

e.g. $f(x) = x^3$ is increasing at every point.

Note that, $dy/dx = 3x^2$.



Note...

1. If a function is invertible it has to be either increasing or decreasing.
2. If a function is continuous, the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.
3. If f is increasing in [a, b] and is continuous then $f(b)$ is the greatest and $f(a)$ is the least value of f in [a, b]. Similarly if f is decreasing in [a, b] then $f(a)$ is the greatest value and $f(b)$ is the least value.

5. (a) ROLLE'S Theorem :

Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point x = c such that $a < c < b$ where $f'(c) = 0$.

(b) LMVT Theorem :

Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

Then there exists at least one point x = c such that $a < c < b$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c.

Note the following : Rolles theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

Note...

Physical Enterpretation of LMVT :

Now $[f(b) - f(a)]$ is the change in the function f as x changes from a to b so that $\dfrac{f(b) - f(a)}{b - a}$ is the average rate of change of the function over the interval [a, b]. Also $f'(c)$ is the actual rate of change of the function for x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.

This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) Application of rolles theorem for isolating the real roots of an equation $f(x) = 0$

Suppose a & b are two real numbers such that :

(i) $f(x)$ & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(ii) $f(a)$ & $f(b)$ have opposite signs.

(iii) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one real root of the equation $f(x) = 0$ between a & b.

8. HOW MAXIMA & MINIMA ARE CLASSIFIED

1. A function $f(x)$ is said to have a local maximum at x = a if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of x = a. Symbolically

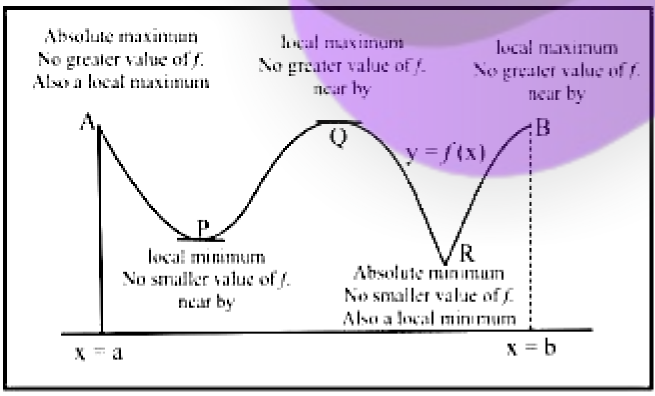
$$\left.\begin{array}{l} f(a) > f(a+h) \\ f(a) > f(a-h) \end{array}\right] \Rightarrow x=a \text{ gives maxima}$$

for a sufficiently small positive h.

Similarly, a function $f(x)$ is said to have a local minimum value at x = b if $f(b)$ is least than every other value assumed by $f(x)$ in the immediate neighbourhood at x = b. Symbolically if

$$\left.\begin{array}{l} f(b) < f(b+h) \\ f(b) < f(b-h) \end{array}\right] \Rightarrow x=b \text{ gives minima for a sufficiently}$$

small positive h.



Note..

(i) The local maximum & local minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) The term 'extremum' is used both for maxima or a minima.

(iii) A local maximum (local minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) A function can have several local maximum & local minimum values & a local minimum value may even be greater than a local maximum value.

(v) Maxima & minima of a continuous function occur alternately & between two consecutive maxima there is a minima & vice versa.

2. A necessary condition for maxima & minima

If $f(x)$ is a maxima or minima at x = c & if $f'(c)$ exists then $f'(c) = 0$.

Note..

(i) The set of values of x for which $f'(x) = 0$ are often called as stationary points. The rate of change of function is zero at a stationary point.

(ii) In case $f'(c)$ does not exist $f(c)$ may be a maxima or a minima & in this case left hand and right hand derivatives are of opposite signs.

(iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval [a, b] are $f(a)$ or $f(b)$ or are given by the values of x which are critical points.

(iv) **Critical points** are those where :

(i) $\frac{dy}{dx} = 0$, if it exists; (ii) or it fails to exist

3. Sufficient condition for extreme values First Derivative Test

$$\left.\begin{matrix} f'(c-h) > 0 \\ f'(c+h) < 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local maxima,}$$

where h is a sufficiently small positive quantity

Similarly $$\left.\begin{matrix} f'(c-h) < 0 \\ f'(c+h) > 0 \end{matrix}\right] \Rightarrow x = c \text{ is a point of local minima,}$$

where h is a sufficiently small positive quantity

Note :– $f'(c)$ in both the cases may or may not exist. If it exists, then $f'(c) = 0$.

Note...

If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either strictly increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f.

4. Use of second order derivative in ascertaining the maxima or minima

(a) $f(c)$ is a minima of the function f, if $f'(c) = 0$ & $f''(c) > 0$.

(b) $f(c)$ is a maxima of the function f, if $f'(c) = 0$ & $f''(c) < 0$.

Note...

If $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaining the maxima or minima.

5. Summary-working rule

First : When possible, draw a figure to illustrate them problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second : Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third : If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

Fourth : Test each values of x for which $f'(x) = 0$ to determine whether it provides a maxima or minima or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0$

$\Rightarrow$ y is minima.

If d^2y/dx^2 is negative when $dy/dx = 0$

$\Rightarrow$ y is maxima.

If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) If $$\frac{dy}{dx} \text{ is } \left.\begin{matrix} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{matrix}\right] \Rightarrow \text{a maxima occurs at } x = x_0$$

But if dy/dx changes sign from negative to zero to positive as x advances through x_0, there is a minima. If dy/dx does not change sign, neither a maxima nor a minima. Such points are called **INFLECTION POINTS**.

Fifth : If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ & $x = b$ for possible extreme values.

Sixth : If the derivative fails to exist at some point, examine this point as possible maxima or minima.

(In general, check at all Critical Points).

Note...

- If the sum of two positive numbers x and y is constant than their product is maximum if they are equal, i.e. $x + y = c$, $x > 0$, $y > 0$, then

  $$xy = \frac{1}{4}\left[(x+y)^2 - (x-y)^2\right]$$

- If the product of two positive numbers is constant then their sum is least if they are equal.

  i.e. $(x + y)^2 = (x - y)^2 + 4xy$

6. Useful formulae of mensuration to remember

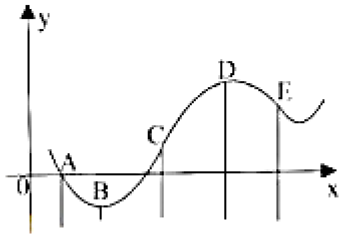
- Volume of a cuboid = lbh.
- Surface area of a cuboid = $2(lb + bh + hl)$.
- Volume of a prism = area of the base × height.
- Lateral surface of a prism = perimeter of the base × height.
- Total surface of a prism = lateral surface + 2 area of the base

  (Note that lateral surfaces of a prism are all rectangles).
- Volume of a pyramid = $\frac{1}{3}$ area of the base × height.
- Curved surface of a pyramid = $\frac{1}{2}$ (perimeter of the base) × slant height.

  (Note that slant surfaces of a pyramid are triangles).
- Volume of a cone = $\frac{1}{3}\pi r^2 h$.
- Curved surface of a cylinder = $2\pi rh$.
- Total surface of a cylinder = $2\pi rh + 2\pi r^2$.
- Volume of a sphere = $\frac{4}{3}\pi r^3$.
- Surface area of a sphere = $4\pi r^2$.
- Area of a circular sector = $\frac{1}{2}r^2\theta$, where θ is in radians.

7. Significance of the sign of 2nd order derivative and points of inflection

The sign of the 2nd order derivative determines the concavity of the curve. Such point such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if :



(i) $\frac{d^2}{dx^2} > 0 \Rightarrow$ concave upwards

(ii) $\frac{d^2}{dx^2} < 0 \Rightarrow$ concave downwards.

At the point of inflection we find that $\frac{d^2}{dx^2} = 0$ and $\frac{d^2}{dx^2}$ changes sign.

Inflection points can also occur if $\frac{d^2}{dx^2}$ fails to exist (but changes its sign). For example, consider the graph of the function defined as,

$$f(x) = \begin{bmatrix} x^{3/5} & \text{for } x \in (-\infty, 1) \\ 2 - x^2 & \text{for } x \in (1, \infty) \end{bmatrix}$$

Note...

The graph below exhibits two critical points one is a point of local maximum (x = c) & the other a point of inflection (x = 0). This implies that not every Critical Point is a point of extrema.