

Chapter : 4. PRINCIPLE OF MATHEMATICAL INDUCTION

Exercise : 4

Question: 1

Solution:

To Prove:

$$1 + 2 + 3 + 4 + \dots + n = \frac{1}{2} n(n + 1)$$

Steps to prove by mathematical induction:

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true

(ii) $P(k + 1)$ is true, whenever $P(k)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

Therefore,

$$\text{Let } P(n): 1 + 2 + 3 + 4 + \dots + n = \frac{1}{2} n(n + 1)$$

Step 1:

$$P(1) = \frac{1}{2} 1(1 + 1) = \frac{1}{2} \times 2 = 1$$

Therefore, $P(1)$ is true

Step 2:

Let $P(k)$ is true Then,

$$P(k): 1 + 2 + 3 + 4 + \dots + k = \frac{1}{2} k(k + 1)$$

Now,

$$1 + 2 + 3 + 4 + \dots + k + (k + 1) = \frac{1}{2} k(k + 1) + (k + 1)$$

$$= (k + 1) \left\{ \frac{1}{2} k + 1 \right\}$$

$$= \frac{1}{2} (k + 1) (k + 2)$$

$$= P(k + 1)$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true

Hence, by the principle of mathematical induction, we have

$$1 + 2 + 3 + 4 + \dots + n = \frac{1}{2} n(n + 1) \text{ for all } n \in \mathbb{N}$$

Hence proved.

Question: 2

Solution:

To Prove:

$$2 + 4 + 6 + 8 + \dots + 2n = n(n + 1)$$

Steps to prove by mathematical induction:

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true

(ii) $P(k+1)$ is true, whenever $P(k)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

Therefore,

Let $P(n)$: $2 + 4 + 6 + 8 + \dots + 2n = n(n+1)$

Step 1:

$$P(1) = 1(1+1) = 1 \times 2 = 2$$

Therefore, $P(1)$ is true

Step 2:

Let $P(k)$ is true Then,

$$P(k): 2 + 4 + 6 + 8 + \dots + 2k = k(k+1)$$

Now,

$$2 + 4 + 6 + 8 + \dots + 2k + 2(k+1) = k(k+1) + 2(k+1)$$

$$= k(k+1) + 2(k+1)$$

$$= (k+1)(k+2)$$

$$= P(k+1)$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true

Hence, by the principle of mathematical induction, we have

$$2 + 4 + 6 + 8 + \dots + 2n = n(n+1) \text{ for all } n \in \mathbb{N}$$

Question: 3

Solution:

To Prove:

$$1 + 3^1 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$$

Steps to prove by mathematical induction:

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true

(ii) $P(k+1)$ is true, whenever $P(k)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

Therefore,

$$\text{Let } P(n): 1 + 3^1 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$$

Step 1:

$$P(1) = \frac{3^1 - 1}{2} = \frac{2}{2} = 1$$

Therefore, $P(1)$ is true

Step 2:

Let $P(k)$ is true Then,

$$P(k): 1 + 3^1 + 3^2 + \dots + 3^{k-1} = \frac{3^k - 1}{2}$$

Now,

$$\begin{aligned}
 1 + 3^1 + 3^2 + \dots + 3^{k-1} + 3^{(k+1)-1} &= \frac{3^{(k)} - 1}{2} + 3^{(k+1)-1} \\
 &= \frac{3^k - 1}{2} + 3^{(k)} \\
 &= 3^{(k)} \left(\frac{1}{2} + 1 \right) - \frac{1}{2} \\
 &= 3^{(k)} \left(\frac{3}{2} \right) - \frac{1}{2} \\
 &= 3^{(k+1)} \left(\frac{1}{2} \right) - \frac{1}{2} \\
 &= \frac{3^{(k+1)} - 1}{2}
 \end{aligned}$$

$$= P(k+1)$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true

Hence, by the principle of mathematical induction, we have

$$1 + 3^1 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2} \text{ for all } n \in \mathbb{N}$$

Question: 4

Solution:

To Prove:

$$2 + 6 + 18 + \dots + 2 \times 3^{n-1} = (3^n - 1)$$

Steps to prove by mathematical induction:

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true

(ii) $P(k+1)$ is true, whenever $P(k)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

Therefore,

$$\text{Let } P(n): 2 + 6 + 18 + \dots + 2 \times 3^{n-1} = (3^n - 1)$$

Step 1:

$$P(1) = 3^1 - 1 = 3 - 1 = 2$$

Therefore, $P(1)$ is true

Step 2:

Let $P(k)$ is true Then,

$$P(k): 2 + 6 + 18 + \dots + 2 \times 3^{k-1} = (3^k - 1)$$

Now,

$$2 + 6 + 18 + \dots + 2 \times 3^{k-1} + 2 \times 3^{k+1-1} = (3^k - 1) + 2 \times 3^k$$

$$= -1 + 3 \times 3^k$$

$$= 3^{k+1} - 1$$

$$= P(k+1)$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true

Hence, by the principle of mathematical induction, we have

$$2 + 6 + 18 + \dots + 2 \times 3^{n-1} = (3^n - 1) \text{ for all } n \in \mathbb{N}$$

Question: 5

Solution:

To Prove:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right)$$

Steps to prove by mathematical induction:

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true

(ii) $P(k+1)$ is true, whenever $P(k)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

Therefore,

$$\text{Let } P(n): \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right)$$

Step 1:

$$P(1) = 1 - \frac{1}{2^1} = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore, $P(1)$ is true

Step 2:

Let $P(k)$ is true Then,

$$P(k): \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$$

Now,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= 1 + \frac{1}{2^k} \left(\frac{1}{2} - 1\right)$$

$$= 1 + \frac{1}{2^k} \left(-\frac{1}{2}\right)$$

$$= 1 - \frac{1}{2^{k+1}}$$

$$= P(k+1)$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true

Hence, by the principle of mathematical induction, we have

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = \left(1 - \frac{1}{2^n}\right) \text{ for all } n \in \mathbb{N}$$

Question: 6

Solution:

To Prove:

$$1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Steps to prove by mathematical induction:

Let $P(n)$ be a statement involving the natural number n such that

(i) $P(1)$ is true

(ii) $P(k+1)$ is true, whenever $P(k)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

Therefore,

$$\text{Let } P(n): 1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Step 1:

$$P(1) = \frac{1(2-1)(2+1)}{3} = \frac{3}{3} = 1$$

Therefore, $P(1)$ is true

Step 2:

Let $P(k)$ is true Then,

$$P(k): 1^2 + 3^2 + 5^2 + 7^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

Now,

$$\begin{aligned} 1^2 + 3^2 + 5^2 + 7^2 + \dots + (2(k+1)-1)^2 &= \frac{k(2k-1)(2k+1)}{3} + (2k+2-1)^2 \\ &= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\ &= (2k+1) \left[\frac{k(2k-1)}{3} + 2k+1 \right] \\ &= (2k+1) \left[\frac{2k^2 - k + 6k + 3}{3} \right] \\ &= (2k+1) \left[\frac{2k^2 + 5k + 3}{3} \right] \\ &= (2k+1) \left[\frac{(k+1)(2k+3)}{3} \right] \text{ (Splitting the middle term)} \\ &= \frac{(k+1)(2k+1)(2k+3)}{3} \end{aligned}$$

$$= P(k+1)$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true

Hence, by the principle of mathematical induction, we have

$$1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} \text{ for all } n \in \mathbb{N}$$

Question: 7

Solution:

To Prove:

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \dots + n \times 2^n = (n-1)2n + 1 + 2$$

Let us prove this question by principle of mathematical induction (PMI)

Let $P(n): 1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + \dots + n \times 2^n$

For $n = 1$

$$\text{LHS} = 1 \times 2 = 2$$

$$\text{RHS} = (1 - 1) \times 2^{(1+1)} + 2$$

$$= 0 + 2 = 2$$

Hence, LHS = RHS

$P(n)$ is true for $n = 1$

Assume $P(k)$ is true

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k = (k - 1) \times 2^{k+1} + 2 \dots (1)$$

We will prove that $P(k + 1)$ is true

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + (k + 1) \times 2^{k+1} = ((k + 1) - 1) \times 2^{(k+1)+1} + 2$$

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + (k + 1) \times 2^{k+1} = (k) \times 2^{k+2} + 2$$

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k + (k + 1) \times 2^{k+1} = (k) \times 2^{k+2} + 2 \dots (2)$$

We have to prove $P(k + 1)$ from $P(k)$, i.e. (2) from (1)

From (1)

$$1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k = (k - 1) \times 2^{k+1} + 2$$

Adding $(k + 1) \times 2^{k+1}$ both sides,

$$(1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k) + (k + 1) \times 2^{k+1} = (k - 1) \times 2^{k+1} + 2 + (k + 1) \times 2^{k+1}$$

$$= k \times 2^{k+1} - 2^{k+1} + 2 + k \times 2^{k+1} + 2^{k+1}$$

$$= 2k \times 2^{k+1} + 2$$

$$= k \times 2^{k+2} + 2$$

$$(1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3 + k \times 2^k) + (k + 1) \times 2^{k+1} = k \times 2^{k+2} + 2$$

which is the same as $P(k + 1)$

Therefore, $P(k + 1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for

where n is a natural number

Put $k = n - 1$

$$(1 \times 2^1 + 2 \times 2^2 + 3 \times 2^3) + n \times 2^n = (n - 1) \times 2^{n+1} + 2$$

Hence proved.

Question: 8

Solution:

To Prove:

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots + 3^n \times 2^{n+1} = \frac{12}{5} (6^n - 1)$$

Let us prove this question by principle of mathematical induction (PMI)

$$\text{Let } P(n): 3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots + 3^n \times 2^{n+1}$$

For $n = 1$

$$\text{LHS} = 3 \times 2^2 = 12$$

$$\text{RHS} = \left(\frac{12}{5}\right) \times (6^1 - 1)$$

$$= \frac{12}{5} \times 5 = 12$$

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots \times + 3^k \times 2^{k+1} = \frac{12}{5}(6^k - 1) \dots\dots(1)$$

We will prove that P(k + 1) is true

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots \times + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1} - 1)$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots \times + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1}) - \frac{12}{5}$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots \times + 3^k \times 2^{k+1} + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1}) - \frac{12}{5} \dots(2)$$

We have to prove P(k + 1) from P(k) ie (2) from (1)

From (1)

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots \times + 3^k \times 2^{k+1} = \frac{12}{5}(6^k - 1)$$

Adding $3^{k+1} \times 2^{k+2}$ both sides

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots + 3^k \times 2^{k+1} + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^k - 1) + 3^{k+1} \times 2^{k+2}$$

$$= \frac{12}{5}(6^k - 1) + 3^k \times 2^k \times 12$$

$$= \frac{12}{5}(6^k - 1) + 6^k \times 12$$

$$= \left(6^k \left(\frac{12}{5} + 12\right) - \frac{12}{5}\right)$$

$$= \left(\frac{72}{5}\right) - \frac{12}{5}$$

$$= \frac{12}{5}(6^{k+1}) - \frac{12}{5}$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots + 3^k \times 2^{k+1} + 3^{k+1} \times 2^{k+2} = \frac{12}{5}(6^{k+1}) - \frac{12}{5}$$

which is the same as P(k + 1)

Therefore, P(k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for

where n is a natural number

Put k = n - 1

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots \times + 3^n \times 2^{n+1} = \frac{12}{5}(6^n) - \frac{12}{5}$$

$$3 \times 2^2 + 3^2 \times 2^3 + 3^3 \times 2^4 + \dots \times 3^n \times 2^{n+1} = \frac{12}{5} (6^n - 1)$$

Hence proved

Question: 9

Solution:

To Prove:

$$\frac{1}{1} + \frac{1}{(1+2)} + \frac{1}{(1+2+3+\dots+n)} = \frac{2n}{(n+1)}$$

Let us prove this question by principle of mathematical induction (PMI)

$$\text{Let } P(n): \frac{1}{1} + \frac{1}{(1+2)} + \frac{1}{(1+2+3+\dots+n)} = \frac{2n}{(n+1)}$$

For $n = 1$

$$\text{LHS} = 1$$

$$\text{RHS} = \frac{2 \times 1}{(1+1)} = 1$$

Hence, LHS = RHS

$P(n)$ is true for $n = 1$

Assume $P(k)$ is true

$$\frac{1}{1} + \frac{1}{(1+2)} + \dots + \frac{1}{(1+2+3+\dots+k)} = \frac{2k}{(k+1)} \dots (1)$$

We will prove that $P(k+1)$ is true

$$\text{RHS} = \frac{2(k+1)}{(k+1+1)} = \frac{2k+2}{k+2}$$

$$\text{LHS} = \frac{1}{1} + \frac{1}{(1+2)} + \dots + \frac{1}{(1+2+3+\dots+(k+1))}$$

$$= \frac{1}{1} + \frac{1}{(1+2)} + \dots + \frac{1}{(1+2+3+\dots+k)} + \frac{1}{(1+2+3+\dots+(k+1))} \quad [\text{Writing the last}]$$

Second term]

$$= \frac{2k}{(k+1)} + \frac{1}{(1+2+3+\dots+(k+1))} \quad [\text{From 1}]$$

$$= \frac{2k}{(k+1)} + \frac{1}{\frac{(k+1) \times (k+2)}{2}}$$

$$\{ 1+2+3+4+\dots+n = \frac{n(n+1)}{2} \text{ put } n = k+1 \}$$

$$= \frac{2k}{(k+1)} + \frac{2}{(k+1) \times (k+2)}$$

$$= \frac{2}{(k+1)} \left(\frac{k}{1} + \frac{1}{k+2} \right)$$

$$= \frac{2}{k+1} \left(\frac{(k+1) \times (k+1)}{k+2} \right)$$

[Taking LCM and simplifying]

$$= \frac{2(k+1)}{(k+2)}$$

= RHS

$$\text{Therefore, } \frac{1}{1} + \frac{1}{(1+2)} + \dots + \frac{1}{(1+2+3+\dots+(k+1))} = \frac{2k+2}{k+2}$$

LHS = RHS

Therefore, P(k+1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for

where n is a natural number

Put k = n - 1

$$\frac{1}{1} + \frac{1}{(1+2)} + \dots + \frac{1}{(1+2+3+\dots+n)} = \frac{2n}{n+1}$$

Hence proved

Question: 10

Solution:

To Prove:

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3n-1) \times (3n+2)} = \frac{n}{(6n+4)}$$

For n = 1

$$\text{LHS} = \frac{1}{2 \times 5} = \frac{1}{10}$$

$$\text{RHS} = \frac{1 \times 1}{(6+4)} = \frac{1}{10}$$

Hence, LHS = RHS

P(n) is true for n = 1

Assume P(k) is true

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} = \frac{k}{(6k+4)} \dots (1)$$

We will prove that P(k+1) is true

$$\text{RHS} = \frac{k+1}{(6(k+1)+4)} = \frac{k+1}{(6k+10)}$$

$$\text{LHS} = \frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)}$$

[Writing the Last second term]

$$= \frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)}$$

$$= \frac{k}{(6k+4)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)} \text{ [Using 1]}$$

$$= \frac{k}{(6k+4)} + \frac{1}{(3k+2) \times (3k+5)}$$

$$= \frac{k}{(6k+4)} + \frac{1}{(3k+2) \times (3k+5)}$$

$$= \frac{1}{(3k+2)} \times \left[\frac{(3k+2) \times (k+1)}{2 \times (3k+5)} \right] \text{ (Taking LCM and simplifying)}$$

$$= \frac{k+1}{(6k+10)}$$

= RHS

$$\text{Therefore, } \frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)} = \frac{k+1}{(6k+10)}$$

LHS = RHS

Therefore, $P(k+1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for

where n is a natural number

Put $k = n - 1$

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots \times + \frac{1}{(3n-1) \times (3n+2)} = \frac{n}{(6n+4)}$$

Hence proved.

Question: 11

Solution:

To Prove:

$$\frac{1}{1 \times 4} + \frac{1}{(4 \times 7)} + \dots + \frac{1}{(3n-2) \times (3n+1)} = \frac{n}{(3n+1)}$$

Let us prove this question by principle of mathematical induction (PMI)

$$\text{Let } P(n): \frac{1}{1 \times 4} + \frac{1}{(4 \times 7)} + \dots + \frac{1}{(3n-2) \times (3n+1)} = \frac{n}{(3n+1)}$$

For $n = 1$

$$\text{LHS} = \frac{1}{1 \times 4} = \frac{1}{4}$$

$$\text{RHS} = \frac{1}{(3+1)} = \frac{1}{4}$$

Hence, LHS = RHS

$P(n)$ is true for $n = 1$

Assume $P(k)$ is true

$$= \frac{1}{1 \times 4} + \frac{1}{(4 \times 7)} + \dots + \frac{1}{(3k-2) \times (3k+1)} = \frac{k}{(3k+1)} \dots (1)$$

We will prove that $P(k+1)$ is true

$$\text{RHS} = \frac{k+1}{(3(k+1)+1)} = \frac{k+1}{(3k+4)}$$

$$\text{LHS} = \frac{1}{1 \times 4} + \frac{1}{(4 \times 7)} + \dots + \frac{1}{(3(k+1)-2) \times (3(k+1)+1)}$$

$$= \frac{1}{1 \times 4} + \frac{1}{(4 \times 7)} + \dots + \frac{1}{(3k-2) \times (3k+1)} + \frac{1}{(3k+1) \times (3k+4)}$$

[Writing the second last term]

$$= \frac{k}{(3k+1)} + \frac{1}{(3k+1) \times (3k+4)} \text{ [Using 1]}$$

$$= \frac{1}{(3k+1)} \left(k + \frac{1}{(3k+4)} \right)$$

$$= \frac{1}{(3k+1)} \left(\frac{3k^2 + 4k + 1}{(3k+4)} \right)$$

$$= \frac{k+1}{(3k+4)}$$

(Splitting the numerator and cancelling the common factor)

= RHS

LHS = RHS

Therefore, $P(k+1)$ is true whenever $P(k)$ is true.

By the principle of mathematical induction, $P(n)$ is true for

where n is a natural number

Hence proved.

Question: 12

Solution:

To Prove:

$$\frac{1}{1 \times 3} + \frac{1}{(3 \times 5)} + \dots + \frac{1}{(2n-1) \times (2n+1)} = \frac{n}{(2n+1)}$$

Let us prove this question by principle of mathematical induction (PMI)

$$\text{Let } P(n): \frac{1}{1 \times 3} + \frac{1}{(3 \times 5)} + \dots + \frac{1}{(2n-1) \times (2n+1)} = \frac{n}{(2n+1)}$$

For $n = 1$

$$\text{LHS} = \frac{1}{1 \times 3} = \frac{1}{3}$$

$$\text{RHS} = \frac{1}{(2+1)} = \frac{1}{3}$$

Hence, $\text{LHS} = \text{RHS}$

$P(n)$ is true for $n = 1$

Assume $P(k)$ is true

$$= \frac{1}{1 \times 3} + \frac{1}{(3 \times 5)} + \dots + \frac{1}{(2k-1) \times (2k+1)} = \frac{k}{(2k+1)} \dots (1)$$

We will prove that $P(k+1)$ is true

$$\text{RHS} = \frac{k+1}{(2(k+1)+1)} = \frac{k+1}{(2k+3)}$$

$$\begin{aligned} \text{LHS} &= \frac{1}{1 \times 3} + \frac{1}{(3 \times 5)} + \dots + \frac{1}{(2(k+1)-1) \times (2(k+1)+1)} \\ &= \frac{1}{1 \times 3} + \frac{1}{(3 \times 5)} + \dots + \frac{1}{(2k-1) \times (2k+1)} + \frac{1}{(2k+1) \times (2k+3)} \end{aligned}$$

[Writing the second last term]

$$= \frac{k}{(2k+1)} + \frac{1}{(2k+1) \times (2k+3)} \text{ [Using 1]}$$

$$= \frac{1}{(2k+1)} \left(k + \frac{1}{(2k+3)} \right)$$

$$= \frac{1}{(2k+1)} \left(\frac{2k^2 + 3k + 1}{(2k+3)} \right)$$

$$= \frac{k+1}{(2k+3)}$$

(Splitting the numerator and cancelling the common factor)

= RHS

LHS = RHS

Therefore, $P(k+1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for

where n is a natural number

Hence proved.

Question: 13

Solution:

To Prove:

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3n-1) \times (3n+2)} = \frac{n}{(6n+4)}$$

For $n = 1$

$$\text{LHS} = \frac{1}{2 \times 5} = \frac{1}{10}$$

$$\text{RHS} = \frac{1 \times 1}{(6+4)} = \frac{1}{10}$$

Hence, LHS = RHS

$P(n)$ is true for $n = 1$

Assume $P(k)$ is true

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} = \frac{k}{(6k+4)} \dots (1)$$

We will prove that $P(k+1)$ is true

$$\text{RHS} = \frac{k+1}{(6(k+1)+4)} = \frac{k+1}{(6k+10)}$$

$$\text{LHS} = \frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)} \quad [\text{Writing the Last second term}]$$

$$= \frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)}$$

$$= \frac{k}{(6k+4)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)} \quad [\text{Using 1}]$$

$$= \frac{k}{(6k+4)} + \frac{1}{(3k+2) \times (3k+5)}$$

$$= \frac{k}{(6k+4)} + \frac{1}{(3k+2) \times (3k+5)}$$

$$= \frac{1}{(3k+2)} \times \left[\frac{(3k+2) \times (k+1)}{2 \times (3k+5)} \right] \quad (\text{Taking LCM and simplifying})$$

$$= \frac{k+1}{(6k+10)}$$

= RHS

$$\text{Therefore, } \frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3k-1) \times (3k+2)} + \frac{1}{(3(k+1)-1) \times (3(k+1)+2)} = \frac{k+1}{(6k+10)}$$

LHS = RHS

Therefore, $P(k+1)$ is true whenever $P(k)$ is true.

By the principle of mathematical induction, $P(n)$ is true for

where n is a natural number

Put $k = n - 1$

$$\frac{1}{2 \times 5} + \frac{1}{(5 \times 8)} + \dots + \frac{1}{(3n-1) \times (3n+2)} = \frac{n}{(6n+4)}$$

Hence proved.

Question: 14

Solution:

To Prove:

$$\left(1 + \frac{3}{1}\right) \times \left(1 + \frac{5}{4}\right) \times \left(1 + \frac{7}{9}\right) \times \dots \times \left\{1 + \frac{2n+1}{n^2}\right\} = (n+1)^2$$

Let us prove this question by principle of mathematical induction (PMI)

$$\text{Let } P(n): \left(1 + \frac{3}{1}\right) \times \left(1 + \frac{5}{4}\right) \times \left(1 + \frac{7}{9}\right) \times \dots \times \left\{1 + \frac{2n+1}{n^2}\right\} = (n+1)^2$$

For $n = 1$

$$\text{LHS} = 1 + \frac{3}{1} = 4$$

$$\text{RHS} = (1+1)^2 = 4$$

Hence, LHS = RHS

$P(n)$ is true for $n = 1$

Assume $P(k)$ is true

$$= \left(1 + \frac{3}{1}\right) \times \left(1 + \frac{5}{4}\right) \times \left(1 + \frac{7}{9}\right) \times \dots \times \left\{1 + \frac{2k+1}{k^2}\right\} = (k+1)^2 \dots (1)$$

We will prove that $P(k+1)$ is true

$$\text{RHS} = ((k+1)+1)^2 = (k+2)^2$$

$$\text{LHS} = \left(1 + \frac{3}{1}\right) \times \left(1 + \frac{5}{4}\right) \times \left(1 + \frac{7}{9}\right) \times \dots \times \left\{1 + \frac{2(k+1)+1}{(k+1)^2}\right\}$$

[Now writing the second last term]

$$= \left(1 + \frac{3}{1}\right) \times \left(1 + \frac{5}{4}\right) \times \left(1 + \frac{7}{9}\right) \times \dots \times \left\{1 + \frac{2k+1}{k^2}\right\} \times \left\{1 + \frac{2(k+1)+1}{(k+1)^2}\right\}$$

$$= (k+1)^2 \times \left\{1 + \frac{2(k+1)+1}{(k+1)^2}\right\} \text{ [Using 1]}$$

$$= (k+1)^2 \times \left\{1 + \frac{(2k+3)}{(k+1)^2}\right\}$$

$$= (k+1)^2 \times \left\{\frac{(k+1)^2 + (2k+3)}{(k+1)^2}\right\}$$

$$= (k+1)^2 + (2k+3)$$

$$= k^2 + 2k + 1 + 2k + 3$$

$$= (k+2)^2$$

$$= \text{RHS}$$

$$\text{LHS} = \text{RHS}$$

Therefore, $P(k+1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for

where n is a natural number

Hence proved.

Question: 15

Solution:

To Prove:

$$\left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \dots \times \left\{1 + \frac{1}{n^1}\right\} = (n + 1)^1$$

Let us prove this question by principle of mathematical induction (PMI)

$$\text{Let } P(n): \left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \dots \times \left\{1 + \frac{1}{n^1}\right\} = (n + 1)^1$$

For $n = 1$

$$\text{LHS} = 1 + \frac{1}{1} = 2$$

$$\text{RHS} = (1 + 1)^1 = 2$$

Hence, LHS = RHS

$P(n)$ is true for $n = 1$

Assume $P(k)$ is true

$$= \left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \dots \times \left\{1 + \frac{1}{n^1}\right\} = (k + 1)^1 \dots (1)$$

We will prove that $P(k + 1)$ is true

$$\text{RHS} = ((k + 1) + 1)^1 = (k + 2)^1$$

$$\text{LHS} = \left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \dots \times \left\{1 + \frac{1}{(k+1)^1}\right\}$$

[Now writing the second last term]

$$= \left(1 + \frac{1}{1}\right) \times \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{3}\right) \times \dots \times \left\{1 + \frac{1}{k^1}\right\} \times \left\{1 + \frac{1}{(k+1)^1}\right\}$$

$$= (k + 1)^1 \times \left\{1 + \frac{1}{(k+1)^1}\right\} \text{ [Using 1] }$$

$$= (k + 1)^1 \times \left\{\frac{(k+1)+1}{(k+1)^1}\right\}$$

$$= (k + 1)^2 \times \left\{\frac{(k+2)^1}{(k+1)^2}\right\}$$

$$= k + 2$$

$$= \text{RHS}$$

$$\text{LHS} = \text{RHS}$$

Therefore, $P(k + 1)$ is true whenever $P(k)$ is true.

By the principle of mathematical induction, $P(n)$ is true for

where n is a natural number

Hence proved.

Question: 16

Solution:

To Prove:

$$n \times (n + 1) \times (n + 2) \text{ is multiple of } 6$$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

$$n \times (n + 1) \times (n + 2) \text{ is multiple of } 6$$

Let $P(n): n \times (n + 1) \times (n + 2)$, which is multiple of 6

For $n = 1$ $P(n)$ is true since $1 \times (1 + 1) \times (1 + 2) = 6$, which is multiple of 6

Assume $P(k)$ is true for some positive integer k , ie,

$$= k \times (k + 1) \times (k + 2) = 6m, \text{ where } m \in \mathbb{N} \dots(1)$$

We will now prove that $P(k + 1)$ is true whenever $P(k)$ is true

Consider ,

$$= (k + 1) \times ((k + 1) + 1) \times ((k + 1) + 2)$$

$$= (k + 1) \times \{k + 2\} \times \{(k + 2) + 1\}$$

$$= [(k + 1) \times (k + 2) \times (k + 2)] + (k + 1) \times (k + 2)$$

$$= [k \times (k + 1) \times (k + 2) + 2 \times (k + 1) \times (k + 2)] + (k + 1) \times (k + 2)$$

$$= [6m + 2 \times (k + 1) \times (k + 2)] + (k + 1) \times (k + 2)$$

$$= 6m + 3 \times (k + 1) \times (k + 2)$$

Now , $(k + 1)$ & $(k + 2)$ are consecutive integers , so their product is even

$$\text{Then , } (k + 1) \times (k + 2) = 2 \times w \text{ (even)}$$

Therefore ,

$$= 6m + 3 \times [2 \times w]$$

$$= 6m + 6 \times w$$

$$= 6(m + w)$$

$$= 6 \times q \text{ where } q = (m + w) \text{ is some natural number}$$

Therefore

$$(k + 1) \times ((k + 1) + 1) \times ((k + 1) + 2) \text{ is multiple of } 6$$

Therefore, $P(k + 1)$ is true whenever $P(k)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all natural numbers, ie, \mathbb{N}

Hence proved.

Question: 17

Solution:

To Prove:

$$x^{2n} - y^{2n} \text{ is divisible by } x + y$$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

$$\text{Let } P(n): x^{2n} - y^{2n} \text{ is divisible by } x + y$$

$$\text{For } n = 1 \text{ } P(n) \text{ is true since } x^{2n} - y^{2n} = x^2 - y^2 = (x + y) \times (x - y)$$

which is divisible by $x + y$

Assume $P(k)$ is true for some positive integer k , ie,

$$= x^{2k} - y^{2k} \text{ is divisible by } x + y$$

$$\text{Let } x^{2k} - y^{2k} = m \times (x + y), \text{ where } m \in \mathbb{N} \dots(1)$$

We will now prove that $P(k + 1)$ is true whenever $P(k)$ is true

Consider ,

$$= x^{2(k+1)} - y^{2(k+1)}$$

$$= x^{2k} \times x^2 - y^{2k} \times y^2$$

$$= x^2(x^{2k} - y^{2k} + y^{2k}) - y^{2k} \times y^2 \text{ [Adding and subtracting } y^{2k} \text{]}$$

$$= x^2(m \times (x + y) + y^{2k}) - y^{2k} \times y^2 \text{ [Using 1]}$$

$$= m \times (x + y)x^2 + y^{2k}x^2 - y^{2k}y^2$$

$$= m \times (x + y)x^2 + y^{2k}(x^2 - y^2)$$

$$= m \times (x + y)x^2 + y^{2k}(x - y)(x + y)$$

$$= (x + y) \{ mx^2 + y^{2k}(x - y) \}, \text{ which is factor of } (x + y)$$

Therefore, $P(k + 1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for all natural numbers ie, N

Hence proved

Question: 18

Solution:

To Prove:

$$x^{2n-1} - 1 \text{ is divisible by } x - 1$$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

$$\text{Let } P(n): x^{2n-1} - 1 \text{ is divisible by } x - 1$$

For $n = 1$

$$P(n) \text{ is true since } x^{2n-1} - 1 = x^{2-1} - 1 = (x - 1)$$

which is divisible by $x - 1$

Assume $P(k)$ is true for some positive integer k , ie,

$$= x^{2k-1} - 1 \text{ is divisible by } x - 1$$

$$\text{Let } x^{2k-1} - 1 = m \times (x - 1), \text{ where } m \in N \dots (1)$$

We will now prove that $P(k + 1)$ is true whenever $P(k)$ is true

Consider ,

$$= x^{2(k+1)-1} - 1$$

$$= x^{2k-1} \times x^2 - 1$$

$$= x^2(x^{2k-1}) - 1$$

$$= x^2(x^{2k-1} - 1 + 1) - 1 \text{ [Adding and subtracting 1]}$$

$$= x^2(m \times (x - 1) + 1) - 1 \text{ [Using 1]}$$

$$= x^2(m \times (x - 1)) + x^2 \times 1 - 1$$

$$= x^2(m \times (x - 1)) + x^2 - 1$$

$$= x^2(m \times (x - 1)) + (x^2 - 1)(x + 1)$$

$$= (x - 1) \{ mx^2 + (x + 1) \}, \text{ which is factor of } (x - 1)$$

Therefore, $P(k + 1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for all natural numbers, ie, N .

Hence proved.

Question: 19

Solution:

To Prove:

$41^n - 14^n$ is a divisible of 27

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let $P(n)$: $41^n - 14^n$ is a divisible of 27

For $n = 1$ $P(n)$ is true since $41^n - 14^n = 41^1 - 14^1 = 27$

which is multiple of 27

Assume $P(k)$ is true for some positive integer k , ie,

$= 41^k - 14^k$ is a divisible of 27

$\therefore 41^k - 14^k = m \times 27$, where $m \in \mathbb{N} \dots(1)$

We will now prove that $P(k + 1)$ is true whenever $P(k)$ is true

Consider ,

$$= 41^{k+1} - 14^{k+1}$$

$$= 41^k \times 41 - 14^k \times 14$$

$$= 41(41^k - 14^k + 14^k) - 14^k \times 14 \text{ [Adding and subtracting } 14^k \text{]}$$

$$= 41(41^k - 14^k) + 41 \times 14^k - 14^k \times 14$$

$$= 41(27m) + 14^k(41 - 14) \text{ [Using 1]}$$

$$= 41(27m) + 14^k(27)$$

$$= 27(41m + 14^k)$$

$$= 27 \times r, \text{ where } r = (41m + 14^k) \text{ is a natural number}$$

Therefore $41^{k+1} - 14^{k+1}$ is divisible of 27

Therefore, $P(k + 1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for all natural numbers, ie, \mathbb{N} .

Hence proved.

Question: 20

Solution:

To Prove:

$4^n + 15n - 1$ is a divisible of 9

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let $P(n)$: $4^n + 15n - 1$ is a divisible of 9

For $n = 1$ $P(n)$ is true since $4^n + 15n - 1 = 4^1 + 15 \times 1 - 1 = 18$

which is divisible of 9

Assume $P(k)$ is true for some positive integer k , ie,

$= 4^k + 15k - 1$ is a divisible of 9

$\therefore 4^k + 15k - 1 = m \times 9$, where $m \in \mathbb{N} \dots(1)$

We will now prove that $P(k+1)$ is true whenever $P(k)$ is true.

Consider ,

$$= 4^{k+1} + 15(k+1) - 1$$

$$= 4^k \times 4 + 15k + 15 - 1$$

$$= 4^k \times 4 + 15k + 14 + (60k + 4) - (60k + 4) \text{ [Adding and subtracting } 60k + 4 \text{]}$$

$$= (4^{k+1} + 60k - 4) + 15k + 14 - (60k - 4)$$

$$= 4(4^k + 15k - 1) + 15k + 14 - (60k - 4)$$

$$= 4(9m) - 45k + 18 \text{ [Using 1]}$$

$$= 4(9m) - 9(5k - 2)$$

$$= 9 [(4m) - (5k - 2)]$$

$$= 9 \times r, \text{ where } r = [(4m) - (5k - 2)] \text{ is a natural number}$$

Therefore $4^k + 15k - 1$ is a divisible of 9

Therefore, $P(k+1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for all natural numbers, ie, N.

Hence proved.

Question: 21

Solution:

To Prove:

$$3^{2n+2} - 8n - 9 \text{ is a divisible of } 8$$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

$$\text{Let } P(n): 3^{2n+2} - 8n - 9 \text{ is a divisible of } 8$$

For $n = 1$ $P(n)$ is true since

$$3^{2n+2} - 8n - 9 = 3^{2+2} - 8 \times 1 - 9 = 81 - 17 = 64$$

which is divisible of 8

Assume $P(k)$ is true for some positive integer k , ie,

$$= 3^{2k+2} - 8k - 9 \text{ is a divisible of } 8$$

$$\therefore 3^{2k+2} - 8k - 9 = m \times 8, \text{ where } m \in \mathbb{N} \dots(1)$$

We will now prove that $P(k+1)$ is true whenever $P(k)$ is true

Consider ,

$$= 3^{2(k+1)+2} - 8(k+1) - 9$$

$$= 3^{2(k+1)} \times 3^2 - 8k - 8 - 9$$

$$= 3^2(3^{2(k+1)} - 8k - 9 + 8k + 9) - 8k - 8 - 9$$

[Adding and subtracting $8k + 9$]

$$= 3^2(3^{2(k+1)} - 8k - 9) + 3^2(8k + 9) - 8k - 17$$

$$= 9(3^{2k+2} - 8k - 9) + 9(8k + 9) - 8k - 17$$

$$= 9(8m) + 72k + 81 - 8k - 17 \text{ [Using 1]}$$

$$= 9(8m) + 64k + 64$$

$$= 8(9m + 8k + 8)$$

$$= 8 \times r, \text{ where } r = 9m + 8k + 8 \text{ is a natural number}$$

Therefore $3^{2k+2} - 8k - 9$ is a divisible of 8

Therefore, $P(k+1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for all natural numbers, ie, N .

Hence proved.

Question: 22

Solution:

To Prove:

$2^{3n} - 1$, which is multiple of 7

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

$2^{3n} - 1$ is multiple of 7

Let $P(n)$: $2^{3n} - 1$, which is multiple of 7

For $n = 1$ $P(n)$ is true since $2^3 - 1 = 8 - 1 = 7$, which is multiple of 7

Assume $P(k)$ is true for some positive integer k , ie,

$$= 2^{3k} - 1 = 7m, \text{ where } m \in N \dots(1)$$

We will now prove that $P(k+1)$ is true whenever $P(k)$ is true

Consider ,

$$= 2^{3(k+1)} - 1$$

$$= 2^{3k} \times 2^3 - 1$$

$$= 2^{3k} \times 2^3 + 2^3 - 2^3 - 1 \text{ [Adding and subtracting } 2^3 \text{]}$$

$$= 2^3(2^{3k} - 1) + 2^3 - 1$$

$$= 2^3(7m) + 2^3 - 1 \text{ [Using 1]}$$

$$= 2^3(7m) + 7$$

$$= 7(2^3m + 1)$$

$$= 7 \times r, \text{ where } r = 2^3m + 1 \text{ is a natural number}$$

Therefore $2^{3n} - 1$ is multiple of 7

Therefore, $P(k+1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for all natural numbers ie, N

Hence proved

Question: 23

Solution:

To Prove:

$$3^n \geq 2^n$$

Let us prove this question by principle of mathematical induction (PMI) for all natural numbers

Let $P(n): 3^n \geq 2^n$

For $n = 1$ $P(n)$ is true since $3^1 \geq 2^1$ i.e. $3 \geq 2$, which is true

Assume $P(k)$ is true for some positive integer k , i.e.,

$$= 3^k \geq 2^k \dots (1)$$

We will now prove that $P(k + 1)$ is true whenever $P(k)$ is true

Consider ,

$$= 3^{(k+1)}$$

$$\therefore 3^{(k+1)} = 3^k \times 3 > 2^k \times 3 \text{ [Using 1]}$$

$$= 3^k \times 3 > 2^k \times 2 \times \frac{3}{2} \text{ [Multiplying and dividing by 2 on RHS]}$$

$$= 3^{k+1} > 2^{k+1} \times \frac{3}{2}$$

$$\text{Now, } 2^{k+1} \times \frac{3}{2} > 2^{k+1}$$

$$\therefore 3^{k+1} > 2^{k+1}$$

Therefore, $P(k + 1)$ is true whenever $P(k)$ is true

By the principle of mathematical induction, $P(n)$ is true for all natural numbers, i.e., N .

Hence proved.