Chapter : 11. APPLICATIONS OF DERIVATIVES

Exercise : 11A

Question: 1

Solution:

Let the side of the square be a

Rate of change of side= $\frac{da}{dt} = 0.2$ cm/s

Perimeter of the square = 4a

Rate of change of perimeter = $4\frac{da}{dt} = 4 \times 0.2$

$$\frac{dP}{dt} = 0.8 \text{ cm/s}$$

Question: 2

Solution:

Let the radius of the circle be r

$$\frac{dr}{dt} = 0.7 \text{cm/s}$$

Circumference of the circle= $2\pi r$

Rate of change of circumference = $2\pi\frac{dr}{dt}$

$= 2 \times 3.14 \times 0.7$

$$\frac{dC}{dt} = 4.4 \text{ cm/s}$$

Question: 3

Solution:

Let the radius of the circle be r

$$\frac{dr}{dt} = 0.3 \text{cm/s}$$

Area of the circle= πr^2

Rate of change of Area = $2\pi r\frac{dr}{dt}$

$= 2 \times 3.14 \times 10 \times 0.3$

$$\frac{dA}{dt} = 18.84 \text{ cm}^2/\text{s}$$

Question: 4

Solution:

Let the side of the square be a

Rate of change of side= $\frac{da}{dt} = 3$ cm/s

Area of the square = a^2

Rate of change of Area = $2a\frac{da}{dt} = 2 \times 10 \times 3$

$$\frac{dA}{dt} = 60 \text{ cm}^2/\text{s}$$

Question: 5

Solution:

Soap bubble will be in the shape of a sphere

Let the radius of the soap bubble be r

$$\frac{dr}{dt} = 0.2\text{cm/s}$$

Surface area of the soap bubble= $4\pi r^2$

Rate of change of Surface area = $8\pi r\frac{dr}{dt}$

$= 8 \times 3.14 \times 7 \times 0.2$

$$\frac{dS}{dt} = 35.2 \text{ cm}^2/\text{s}$$

Question: 6

Solution:

Soap bubble will be in the shape of a sphere

Let the radius of the soap bubble be r

$$\frac{dr}{dt} = 0.5\text{cm/s}$$

Volume of the soap bubble= $\frac{4}{3}\pi r^3$

Rate of change of Volume = $4\pi r^2\frac{dr}{dt}$

$= 4 \times 3.14 \times 1^2 \times 0.5$

$$\frac{dV}{dt} = 6.28 \text{ cm}^3/\text{s}$$

Question: 7

Solution:

Let the radius of the balloon be r

Let the volume of the spherical balloon be V

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$25\ cm^3/s = 4 \times \pi \times 5^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi}$

Surface area of the bubble $= 4\pi r^2$

Rate of change of Surface area $= 8\pi r \frac{dr}{dt}$

$= 8 \times \pi \times 5 \times \frac{1}{4\pi}$

$\frac{dS}{dt} = 10\ cm^2/s$

Question: 8

Solution:

When we pump a balloon its volume changes.

Let the radius of the balloon be r

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$900\ cm^3/s = 4 \times \pi \times 15^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{900}{4 \times 3.14 \times 225}$

$\frac{dr}{dt} = 0.32\ cm/s$

Question: 9

Solution:

Let the volume of the water tank be V

$V = l \times b \times h$

$V = 25 \times 40 \times h$

$\frac{dV}{dt} = 1000 \times \frac{dh}{dt}$

$500 = 1000 \times \frac{dh}{dt}$

$\frac{dh}{dt} = 0.5\ m/min$

Question: 10

Solution:

Let the radius of the circle be r

$\frac{dr}{dt} = 3.5\ cm/s$

Area of the circle=πr^2

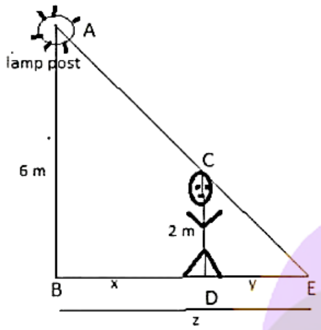
Rate of change of Area $= 2\pi r \frac{dr}{dt}$

$= 2 \times 3.14 \times 7.5 \times 3.5$

$= 165\ cm^2/s$

Question: 11

Solution:



ABE and CDE are similar triangles.

So,

$$\frac{AB}{BE} = \frac{CD}{DE}$$

$$\frac{0.006}{x+y} = \frac{0.002}{y}$$

$$6y = 2(x+y)$$

$$6\frac{dy}{dt} = 2(\frac{dx}{dt} + \frac{dy}{dt})$$

$$6\frac{dy}{dt} = 2(5 + \frac{dy}{dt})$$

$$6\frac{dy}{dt} = 10 + 2\frac{dy}{dt}$$

$$4\frac{dy}{dt} = 10$$

$$\frac{dy}{dt} = 2.5km/h$$

Question: 12

Solution:

Let the volume of the cone be V

$$\frac{dV}{dt} = 1.5\ cm^3/s$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi 5^2 h$$

$V = \frac{25}{3}\pi h$

$\frac{dV}{dt} = \frac{25}{3}\pi\frac{dh}{dt}$

$\frac{15}{10} = \frac{25}{3}\pi\frac{dh}{dt}$

Question: 13

Solution:

$h = \frac{1}{6}r$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi(6h)^2 h$

$V = 12\pi h^3$

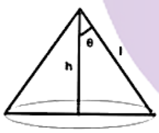
$\frac{dV}{dt} = 36\pi h^2\frac{dh}{dt}$

$18 = 36 \times 9 \times \pi \times \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{18\pi}$ cm/s

Question: 14

Solution:



Let the volume of the cone be V

$\frac{dV}{dt} = 4cm^3/s$

$V = \frac{1}{3}\pi r^2 h$

$\cos Q = \frac{h}{l} = \cos 120 = \cos(180 - 60) = -\frac{1}{2}$

$\sin Q = \frac{r}{l} = \sin 120 = \sin(180 - 60) = \sin 60 = \frac{\sqrt{3}}{2}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{\sqrt{3}}{2}l\right)^2 (-\frac{1}{2}l)$

$V = -\frac{3}{24}\pi l^3$

$\frac{dV}{dt} = -\frac{9}{24}\pi l^2 \frac{dl}{dt}$

$4 = -\frac{3}{8}\pi 3^2 \frac{dl}{dt}$

$-\frac{32}{27\pi}\ \text{cm/s} = \frac{dl}{dt}$

Question: 15

Solution:

$\frac{dV}{dt} = 15\ \text{mL/s}$

$\frac{d}{dt}(\pi r^2 h) = 15$

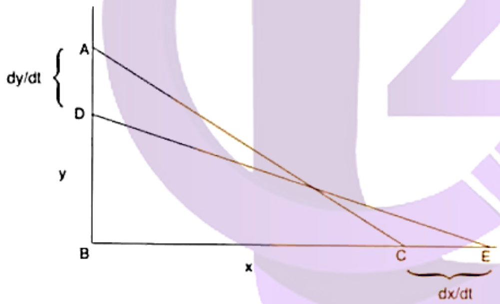
$\frac{d}{dt}(\pi 7^2 h) = 15$

$49\pi \frac{dh}{dt} = 15$

$\frac{dh}{dt} = \frac{15}{49\pi}$

Question: 16

Solution:



Let the original ladder be AC and the pulled ladder be DE

Let AB=y and BC=x

Applying Pythagoras Theorem in ABC

$x^2 + y^2 = 13^2$ ...(1)

$5^2 + y^2 = 13^2$

$y = 12\text{cm}$

Differentiating both sides of eqn (1) wrt to t

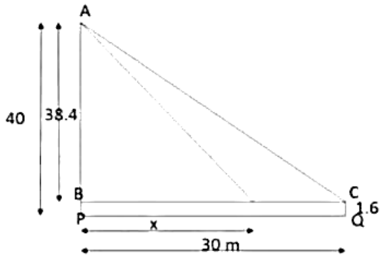
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$5.2 + 12\frac{dy}{dt} = 0$

$$\frac{dy}{dt} = -\frac{10}{12} = -\frac{5}{6}\text{cm/s}$$

Question: 17

Solution:



$$\frac{dx}{dt} = -2\ \text{cm/s}$$

$$\tan Q = \frac{38.4}{x}$$

$$Q = \tan^{-1}\frac{38.4}{x}$$

$$\frac{dQ}{dt} = \frac{1}{1+\frac{38.4^2}{x^2}}\left(-\frac{1}{x^2}\right).38.4$$

$$\frac{dQ}{dt} = \frac{x^2}{x^2+1474.56}\left(-\frac{1}{x^2}\right).38.4$$

$$\frac{dQ}{dt} = -\frac{1}{30^2+1474.56}.38.4.\frac{dx}{dt}$$

$$\frac{dQ}{dt} = -\frac{1}{30^2+1474.56}.38.4\times 2$$

$$\frac{dQ}{dt} = -0.032\ \text{radian/second}$$

Question: 18

Solution:

ATQ,

$$\frac{dx}{dt} = 2\frac{d}{dt}(\sin x)$$

$$\frac{dx}{dt} = 2\cos x\frac{dx}{dt}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

Question: 19

Solution:

$\frac{dr}{dt} = 10\ m/s$

$S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi.15.10$

$\frac{dS}{dt} = 1200\pi\ cm^2/s$

Question: 20

Solution:

$\frac{da}{dt} = 5cm/s$

$V = a^3$

$\frac{dV}{dt} = 3a^2\frac{da}{dt}$

$\frac{dV}{dt} = 3.10^2.5$

$\frac{dV}{dt} = 1500\ cm^3/s$

Question: 21

Solution:

$\frac{da}{dt} = 2cm/s$

$A = \frac{\sqrt{3}}{4}a^2$

$\frac{dA}{dt} = \frac{\sqrt{3}}{4}2a\frac{da}{dt}$

$\frac{dA}{dt} = \frac{\sqrt{3}}{2}.10.2$

$\frac{dA}{dt} = 10\sqrt{3}\ cm^2/s$

Exercise : 11B

Question: 1

Solution:

Let $y = \sqrt{x}$.

Let $x = 36$ and $\Delta x = 1$.

As $y = \sqrt{x}$.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{36}}.1$

$\Rightarrow \Delta y = \frac{1}{12}$

$\therefore \Delta y = 0.08$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.08 = \sqrt{36+1} - \sqrt{36}$

$\Rightarrow 0.08 = \sqrt{37} - 6$

$\Rightarrow \sqrt{37} = 6.08$

Question: 2

Using diffe

Solution:

Let $y = \sqrt[3]{x}$.

Let $x = 27$ and $\Delta x = 2$.

As $y = \sqrt[3]{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{\frac{-2}{3}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{3}x^{\frac{-2}{3}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{3}27^{\frac{-2}{3}}.2$

$\Rightarrow \Delta y = \frac{2}{27}$

$\therefore \Delta y = 0.074$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.074 = \sqrt[3]{27+2} - \sqrt[3]{27}$

$\Rightarrow 0.074 = \sqrt[3]{29} - 3$

$\Rightarrow \sqrt[3]{29} = 3.074$

Question: 3

Solution:

Let $y = \sqrt{x}$.

Let $x = 25$ and $\Delta x = 2$.

As $y = \sqrt{x}$.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{25}}.2$

$\Rightarrow \Delta y = \frac{1}{5}$

$\therefore \Delta y = 0.2$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.2 = \sqrt{25+2} - \sqrt{25}$

$\Rightarrow 0.2 = \sqrt{27} - 5$

$\Rightarrow \sqrt{27} = 5.2$

Question: 4

Solution:

Let $y = \sqrt{x}$.

Let $x = 0.25$ and $\Delta x = -0.01$.

As $y = \sqrt{x}$.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{0.25}}.(-0.01)$

$\Rightarrow \Delta y = -0.01$

$\therefore \Delta y = -0.01$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore -0.01 = \sqrt{0.25-0.01} - \sqrt{0.25}$

$\Rightarrow -0.01 = \sqrt{0.24} - 0.5$

$\Rightarrow \sqrt{0.24} = 0.49$

Question: 5

Solution:

Let $y = \sqrt{x}$.

Let $x = 19$ and $\Delta x = 0.5$.

As $y = \sqrt{x}$.

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{2\sqrt{x}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{2\sqrt{49}}.0.5$

$\Rightarrow \Delta y = \frac{0.5}{14}$

$\therefore \Delta y = 0.0357$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.0357 = \sqrt{49 + 0.5} - \sqrt{49}$

$\Rightarrow 0.0357 = \sqrt{49.5} - 7$

$\Rightarrow \sqrt{49.5} = 7.0357.$

Question: 6

Solution:

Let $y = \sqrt[4]{x}$.

Let $x = 16$ and $\Delta x = 1$.

As $y = \sqrt[4]{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{4}x^{\frac{-3}{4}}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{4}x^{\frac{-3}{4}}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{4}16^{\frac{-3}{4}}.(-1)$

$\Rightarrow \Delta y = \frac{-1}{32}$

$\therefore \Delta y = -0.03125$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore -0.03125 = \sqrt[4]{16-1} - \sqrt[4]{16}$

$\Rightarrow -0.03125 = \sqrt[4]{15} - 2$

$\Rightarrow \sqrt[4]{15} = 1.96875$

Question: 7

Solution:

Let

$y = \frac{1}{x^2}$

Let x =2 and $\Delta x = 0.002$.

As $y = \frac{1}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{x^3}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{-2}{x^3}.\Delta x$

$\Rightarrow \Delta y = \frac{-2}{8}.(0.002)$

$\Rightarrow \Delta y = \frac{-0.5}{1000}$

$\therefore \Delta y = -0.0005$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore -0.0005 = \frac{1}{(2.002)^2} - \frac{1}{2^2}$

$\Rightarrow -0.0005 = \frac{1}{(2.002)^2} - 0.25$

$\Rightarrow \frac{1}{(2.002)^2} = 0.2495$

Question: 8

Solution:

Let $y = \log_e x$

Let x =10 and $\Delta x = 0.02$.

As $y = \log_e x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx}\Delta x$

$\therefore \Delta y = \frac{1}{x}.\Delta x$

$\Rightarrow \Delta y = \frac{1}{10}.(0.02)$

$\Rightarrow \Delta y = \frac{0.02}{10}$

$\therefore \Delta y = 0.002$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.002 = \log_e(10+0.02) - \log_e(10)$

$\Rightarrow 0.002 = \log_e(10.02) - 2.3026$

$\Rightarrow \log_e(10.02) = 2.3046.$

Question: 9

Solution:

Let $y = \log_{10} x$

$\therefore y = \frac{\log_e x}{\log_e 10}$

$\therefore y = 0.4343 \log_e x$

Let x =4 and $\Delta x = 0.04$.

As $y = 0.4343 \log_e x$

$\Rightarrow \frac{dy}{dx} = \frac{0.4343}{x}$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$

$\therefore \Delta y = \frac{0.4343}{x} . \Delta x$

$\Rightarrow \Delta y = \frac{0.4343}{4} . (0.04)$

$\Rightarrow \Delta y = \frac{0.017372}{4}$

$\therefore \Delta y = 0.004343$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.004343 = \log_e(4+0.04) - \log_e(4)$

$\Rightarrow 0.004343 = \log_e(4.04) - 0.6021$

$\Rightarrow \log_e(4.04) = 0.606443.$

Question: 10

Solution:

Let y = cosx

Let x =60° and $\Delta x = 1°$.

As y =cos x

$\Rightarrow \frac{dy}{dx} = \sin x$

We, know

$\Rightarrow \Delta y = \frac{dy}{dx} \Delta x$

$\therefore \Delta y = \sin x . \Delta x$

$\Rightarrow \Delta y = \sin(60)(0.01745)$

$\Rightarrow \Delta y = (0.86603)(0.01745$

$\therefore \Delta y = 0.01511$

Also,

$\Delta y = f(x+\Delta x)-f(x)$

$\therefore 0.01511 = \cos(60+1) - \cos(60)$

$\Rightarrow 0.01511 = \cos 61° - 0.5$

$\Rightarrow \cos 61° = 0.48489$

Question: 11

Solution:

Given x is π /2

Value of π is 22/7

22/14 is π /2

Hence there will be no change.

Question: 12

Solution:

Let the radius of the plate 10cm.

Radius increases by 2% by heating

∴ After increment = $\frac{2}{100} \times 10 = 0.2$

Change in radius dr = 0.2

∴ New radius = 10+0.2 = 10.2cm

Area of circular plate = $A=\pi r^2$

∴ Change in Area = $\frac{dA}{dr}$

$\Rightarrow \frac{dA}{dr} = 2\pi r dr$

$\Rightarrow \frac{dA}{dr} = 2 \times \pi \times 10 \times 0.2$

$\Rightarrow \frac{dA}{dr} = 4\pi \text{ cm}^2$

Question: 13

Solution:

The formula for time period -

$\therefore T = 2\pi\sqrt{\frac{l}{g}}$

Here 2,π,g have no dimensions. So we can eliminate them.

Now $\frac{\Delta T}{T} = \frac{1}{2} \times \frac{\Delta L}{L}$

By representing in percentage error

$\Rightarrow \frac{\Delta T}{T} \times 100\% = \frac{1}{2} \times \frac{\Delta L}{L} \times 100\%$

$\Rightarrow \frac{\Delta T}{T} \times 100\% = \frac{1}{2} \times \frac{\Delta L}{L} \times 100\%$

$\Rightarrow \frac{\Delta T}{T}\% = \frac{1}{2} \times 2\%$

$\Rightarrow \frac{\Delta T}{T}\% = 1\%$

Hence the time period becomes 1 %.

Question: 14

Solution:

Given: $pv^{1/4} = k$

%decrease in the volume = 1/2%s

$\therefore \frac{\Delta V}{V} \times 100 = \frac{-1}{2}$

$pv^{1/4} = k$

taking log on both sides

$\log[pv^{1/4}] = \log a$

$\log P + 1.4\log V = \log a$

Differentiating both the sides we get

$\Rightarrow \frac{1}{p} dP + \frac{1.4}{V} dV = 0$

$\Rightarrow \frac{dP}{P} + 1.4\frac{dV}{V} = 0$

Multiplying both sides by 100.

$\Rightarrow \frac{dP}{P} \times 100 + 1.4 \times \frac{dV}{V} \times 100 = 0$

$\Rightarrow \frac{dP}{P} \times 100 + 1.4\left(\frac{-1}{2}\right) = 0$

$\Rightarrow \frac{dP}{P} \times 100 = 0.7$

%error in P = 0.7%.

Question: 15

The radius of a s

Solution:

Decrease in radius = dr = 10-9.8

$\therefore$ dr = 0.2

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^3$

Change in volume = $dV = 4\pi r^2 dr$

$\therefore dV = 4\pi(10)^2 \times 0.2$

$\Rightarrow dV = 80\pi \text{ cm}^3$

Surface area of the sphere is given by = $A = 4\pi r^2$

Change in volume = $dA = 8\pi r dr$

$\therefore dA = 8\pi \times 10 \times 0.2$

$\therefore dA = 16\pi$.

Question: 16

Solution:

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^2$

Change in volume = $dV = 4\pi r^2 dr$

Given: $\Delta r = 0.1$

$\Rightarrow \Delta r.\frac{dV}{dr} = 4\pi r^2 \Delta r$

$\Rightarrow \Delta V = 4\pi r^2 \Delta r$

Percentage error

$\Rightarrow \frac{\Delta V}{V} = \frac{4\pi r^2}{\frac{4\pi r^3}{3}} \times 0.1$

$= 0.3\%$

Question: 17

Solution:

Let d be the diameter r be the radius and V be the volume of Sphere

Volume of the sphere is given by = $V = \frac{4}{3}\pi r^2$

$\Rightarrow V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{1}{6}\pi D^3$

Let Δd be the error in d and the corresponding error in V be ΔV.

$\therefore \Delta V = \frac{dV}{dd}\Delta d = \frac{1}{2}\pi d^2 \Delta D$

$\therefore \frac{\Delta V}{V} = \frac{\frac{1}{2}\pi d^2 \Delta D}{\frac{1}{6}\pi D^3} = 3\frac{\Delta D}{D}$

Hence Proved

Exercise : 11C

Question: 1

Solution:

Condition (1):

Since, $f(x)=x^2$ is a polynomial and we know every polynomial function is continuous for all $x \epsilon R$.

$\Rightarrow f(x)=x^2$ is continuous on [-1,1].

Condition (2):

Here, $f'(x)=2x$ which exist in [-1,1].

So, $f(x)=x^2$ is differentiable on (-1,1).

Condition (3):

Here, $f(-1)=(-1)^2=1$

And $f(1)=1^1=1$

i.e. $f(-1)=f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(-1,1)$ such that $f'(c)=0$

i.e. $2c=0$

i.e. $c=0$

Value of $c=0\epsilon(-1,1)$

Thus, Rolle's theorem is satisfied.

Question: 2

Solution:

Condition (1):

Since, $f(x)=x^2-x-12$ is a polynomial and we know every polynomial function is continuous for all $x\epsilon R$.

$\Rightarrow f(x)= x^2-x-12$ is continuous on [-3,4].

Condition (2):

Here, $f'(x)=2x-1$ which exist in [-3,4].

So, $f(x)= x^2-x-12$ is differentiable on (-3,4).

Condition (3):

Here, $f(-3)=(-3)^2-3-12=0$

And $f(4)=4^2-4-12=0$

i.e. $f(-3)=f(4)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(-3,4)$ such that $f'(c)=0$

i.e. $2c-1=0$

i.e. $c = \frac{1}{2}$

Value of $c = \frac{1}{2} \epsilon(-3,4)$

Thus, Rolle's theorem is satisfied.

Question: 3

Solution:

Condition (1):

Since, $f(x)=x^2-5x+6$ is a polynomial and we know every polynomial function is continuous for all $x\epsilon R$.

$\Rightarrow f(x)= x^2-5x+6$ is continuous on [2,3].

Condition (2):

Here, $f'(x)=2x-5$ which exist in [2,3].

So, $f(x)= x^2-5x+6$ is differentiable on (2,3).

Condition (3):

Here, $f(2)=2^2-5\times2+6=0$

And $f(3)= 3^2-5\times3+6=0$

i.e. $f(2)=f(3)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(2,3)$ such that $f'(c)=0$

i.e. $2c-5=0$

i.e. $c = \frac{5}{2}$

Value of $c = \frac{5}{2} \in (2,3)$

Thus, Rolle's theorem is satisfied.

Question: 4

Solution:

Condition (1):

Since, $f(x)= x^2-5x+6$ is a polynomial and we know every polynomial function is continuous for all $x \in R$.

⇒ $f(x)= x^2-5x+6$ is continuous on [-3,6].

Condition (2):

Here, $f'(x)=2x-5$ which exist in [-3,6].

So, $f(x)= x^2-5x+6$ is differentiable on (-3,6).

Condition (3):

Here, $f(-3)=(-3)^2-5\times(-3)+6=30$

And $f(6)= 6^2-5\times6+6=12$

i.e. $f(-3) \neq f(6)$

Conditions (3) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

Question: 5

Solution:

Condition (1):

Since, $f(x)=x^2-4x+3$ is a polynomial and we know every polynomial function is continuous for all $x \in R$.

⇒ $f(x)=x^2-4x+3$ is continuous on [1,3].

Condition (2):

Here, $f'(x)=2x-4$ which exist in [1,3].

So, $f(x)=x^2-4x+3$ is differentiable on (1,3).

Condition (3):

Here, $f(1)=(1)^2-4(1)+3=0$

And $f(3)= (3)^2-4(3)+3=0$

i.e. $f(1)=f(3)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (1,3)$ such that $f'(c)=0$

i.e. $2c-4=0$

i.e. $c=2$

Value of $c=2 \in (1,3)$

Thus, Rolle's theorem is satisfied.

Question: 6

Solution:

Condition (1):

Since, $f(x)=x(x-4)^2$ is a polynomial and we know every polynomial function is continuous for all $x \epsilon R$.

$\Rightarrow f(x)= x(x-4)^2$ is continuous on [0,4].

Condition (2):

Here, $f'(x)= (x-4)^2+2x(x-4)$ which exist in [0,4].

So, $f(x)= x(x-4)^2$ is differentiable on (0,4).

Condition (3):

Here, $f(0)=0(0-4)^2=0$

And $f(4)= 4(4-4)^2=0$

i.e. f(0)=f(4)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(0,4)$ such that $f'(c)=0$

i.e. $(c-4)^2+2c(c-4)=0$

i.e. $(c-4)(3c-4)=0$

i.e. c=4 or c=3÷4

Value of $c = \frac{3}{4} \epsilon (0,4)$

Thus, Rolle's theorem is satisfied.

Question: 7

Solution:

Condition (1):

Since, $f(x)=x^3- 7x^2+16x-12$ is a polynomial and we know every polynomial function is continuous for all $x \epsilon R$.

$\Rightarrow f(x)= x^3- 7x^2+16x-12$ is continuous on [2,3].

Condition (2):

Here, $f'(x)=3x^2-14x+16$ which exist in [2,3].

So, $f(x)= x^3- 7x^2+16x-12$ is differentiable on (2,3).

Condition (3):

Here, $f(2)= 2^3- 7(2)^2+16(2)-12=0$

And $f(3)= 3^3- 7(3)^2+16(3)-12=0$

i.e. f(2)=f(3)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(2,3)$ such that $f'(c)=0$

i.e. $3c^2-14c+16=0$

i.e. $(c-2)(3c-7)=0$

i.e. c=2 or c=7÷3

Value of $c = \frac{7}{3} \epsilon (2,3)$

Thus, Rolle's theorem is satisfied.

Question: 8

Solution:

Condition (1):

Since, $f(x)= x^3+3x^2-24x-80$ is a polynomial and we know every polynomial function is continuous for all $x \epsilon R$.

⇒ $f(x)= x^3+3x^2-24x-80$ is continuous on [-4,5].

Condition (2):

Here, $f'(x)= 3x^2+6x-24$ which exist in [-4,5].

So, $f(x)= x^3+3x^2-24x-80$ is differentiable on (-4,5).

Condition (3):

Here, $f(-4)= (-4)^3+3(-4)^2-24(-4)-80=0$

And $f(5)= (5)^3+3(5)^2-24(5)-80=0$

i.e. f(-4)=f(5)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \epsilon (-4,5)$ such that f'(c)=0

i.e. $3c^2+6c-24=0$

i.e. c=-4 or c=2

Value of c=2 ϵ(-4,5)

Thus, Rolle's theorem is satisfied.

Question: 9

Solution:

Condition (1):

Since, f(x)=(x-1)(x-2)(x-3) is a polynomial and we know every polynomial function is continuous for all $x \epsilon R$.

⇒ f(x)= (x-1)(x-2)(x-3) is continuous on [1,3].

Condition (2):

Here, f'(x)= (x-2)(x-3)+ (x-1)(x-3)+ (x-1)(x-2) which exist in [1,3].

So, f(x)= (x-1)(x-2)(x-3) is differentiable on (1,3).

Condition (3):

Here, f(1)= (1-1)(1-2)(1-3) =0

And f(3)= (3-1)(3-2)(3-3) =0

i.e. f(1)=f(3)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \epsilon (1,3)$ such that f'(c)=0 i.e. $(c-2)(c-3)+(c-1)(c-3)+(c-1)(c-2)=0$

i.e. $(c-3)(2c-3)+(c-1)(c-2)=0$

i.e. $(2c^2-9c+9)+(c^2-3c+2)=0$

i.e. $3c^2-12c+11=0$

i.e. $c = \frac{12 \pm \sqrt{12}}{[illegible]}$

i.e. c=2.58 or c=1.42

Value of $c=1.42 \in (1,3)$ and $c=2.58 \in (1,3)$

Thus, Rolle's theorem is satisfied.

Question: 10

Solution:

Condition (1):

Since, $f(x)=(x-1)(x-2)^2$ is a polynomial and we know every polynomial function is continuous for all $x \in R$.

$\Rightarrow f(x)=(x-1)(x-2)^2$ is continuous on [1,2].

Condition (2):

Here, $f'(x)=(x-2)^2+2(x-1)(x-2)$ which exist in [1,2].

So, $f(x)=(x-1)(x-2)^2$ is differentiable on (1,2).

Condition (3):

Here, $f(1)=(1-1)(1-2)^2=0$

And $f(2)=(2-1)(2-2)^2=0$

i.e. $f(1)=f(2)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (1,2)$ such that $f'(c)=0$

i.e. $(c-2)^2+2(c-1)(c-2)=0$

$(3c-4)(c-2)=0$

i.e. $c=2$ or $c=4 \div 3$

Value of $c = \frac{4}{3} = 1.33 \in (1,2)$

Thus, Rolle's theorem is satisfied.

Question: 11

Solution:

Condition (1):

Since, $f(x)=(x-2)^4(x-3)^3$ is a polynomial and we know every polynomial function is continuous for all $x \in R$.

$\Rightarrow f(x)=(x-2)^4(x-3)^3$ is continuous on [2,3].

Condition (2):

Here, $f'(x)=4(x-2)^3(x-3)^3+3(x-2)^4(x-3)^2$ which exist in [2,3].

So, $f(x)=(x-2)^4(x-3)^3$ is differentiable on (2,3).

Condition (3):

Here, $f(2)=(2-2)^4(2-3)^3=0$

And $f(3)=(3-2)^4(3-3)^3=0$

i.e. f(2)=f(3)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∈(2,3) such that f'(c)=0

i.e. $4(c-2)^3(c-3)^3+3(c-2)^4(c-3)^2=0$

$(c-2)^3(c-3)^2(7c-18)=0$

i.e. c=2 or c=3 or c=18÷7

Value of $c=\frac{18}{7}=2.57\in(2,3)$

Thus, Rolle's theorem is satisfied.

Question: 12

Solution:

Condition (1):

Since, $f(x)=\sqrt{1-x^2}$ is a polynomial and we know every polynomial function is continuous for all x∈R.

⇒ $f(x)=\sqrt{1-x^2}$ is continuous on [-1,1].

Condition (2):

Here, $f'(x)=-\frac{x}{\sqrt{1-x^2}}$ which exist in [-1,1].

So, $f(x)=\sqrt{1-x^2}$ is differentiable on (-1,1).

Condition (3):

Here, $f(-1)=\sqrt{1-(-1)^2}=0$

And $f(1)=\sqrt{1-1^2}=0$

i.e. $f(-1)=f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∈(-1,1) such that f'(c)=0

i.e. $-\frac{c}{\sqrt{1-c^2}}=0$

i.e. c=0

Value of c=0∈(-1,1)

Thus, Rolle's theorem is satisfied.

Question: 3

Solution:

Condition (1):

Since, f(x)=cosx is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)=cosx is continuous on $[-\frac{\pi}{2},\frac{\pi}{2}]$.

Condition (2):

Here, f'(x)=-sinx which exist in $[-\frac{\pi}{2},\frac{\pi}{2}]$.

So, f(x)=cosx is differentiable on $(-\frac{\pi}{2},\frac{\pi}{2})$.

Condition (3):

Here, $f\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$

And $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

i.e. $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(-\frac{\pi}{2},\frac{\pi}{2})$ such that f'(c)=0

i.e. -sinc=0

i.e. c=0

Value of $c = 0\epsilon(-\frac{\pi}{2},\frac{\pi}{2})$

Thus, Rolle's theorem is satisfied.

Question: 14

Solution:

Condition (1):

Since, f(x)=cos2x is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)= cos2x is continuous on [0,π].

Condition (2):

Here, f'(x)= -2sin2x which exist in [0,π].

So, f(x)=cos2x is differentiable on (0,π).

Condition (3):

Here, f(0)=cos0=1

And f(π)=cos2π=1

i.e. f(0)=f(π)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one cϵ(0,π) such that f'(c)=0

i.e. -2sin2c =0

i.e. 2c=π

i.e. $c = \frac{\pi}{2}$

Value of $c = \frac{\pi}{2}\epsilon(0,\pi)$

Thus, Rolle's theorem is satisfied.

Question: 15

Verify Rolle's th

Solution:

Condition (1):

Since, f(x)=sin3x is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)= sin3x is continuous on [0,π].

Condition (2):

Here, f'(x)= 3cos3x which exist in [0,π].

So, f(x)= sin3x is differentiable on (0,π).

Condition (3):

Here, f(0)=sin0=0

And f(π)=sin3π=0

i.e. f(0)=f(π)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∈(0,π) such that f'(c)=0

i.e. 3cos3c =0

i.e. $3c = \frac{\pi}{2}$

i.e. $c = \frac{\pi}{6}$

Value of $c = \frac{\pi}{6} \in (0, \pi)$

Thus, Rolle's theorem is satisfied.

Question: 16

Solution:

Condition (1):

Since, f(x)=sinx+cosx is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x)= sinx+cosx is continuous on $[0, \frac{\pi}{2}]$.

Condition (2):

Here, f'(x)= cosx-sinx which exist in $[0, \frac{\pi}{2}]$.

So, f(x)= sinx+cosx is differentiable on $(0, \frac{\pi}{2})$

Condition (3):

Here, f(0)=sin0+cos0=1

And $f(\frac{\pi}{2}) = \sin\left(\frac{\pi}{2}\right) + \cos(\frac{\pi}{2}) = 1$

i.e. $f(0) = f(\frac{\pi}{2})$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c \in (0, \frac{\pi}{2})$ such that f'(c)=0

i.e. cosc-sinc =0

i.e. $c = \frac{\pi}{4}$

Value of $c = \frac{\pi}{4} \epsilon (0, \frac{\pi}{2})$

Thus, Rolle's theorem is satisfied.

Question: 17

Solution:

Condition (1):

Since, $f(x)=e^{-x}\sin x$ is a combination of exponential and trigonometric function which is continuous.

$\Rightarrow f(x)= e^{-x}\sin x$ is continuous on $[0,\pi]$.

Condition (2):

Here, $f'(x)= e^{-x}(\cos x - \sin x)$ which exist in $[0,\pi]$.

So, $f(x)= e^{-x}\sin x$ is differentiable on $(0,\pi)$

Condition (3):

Here, $f(0)= e^{-0}\sin 0=0$

And $f(\pi)= e^{-\pi}\sin\pi =0$

i.e. $f(0)=f(\pi)$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(0,\pi)$ such that $f'(c)=0$

i.e. $e^{-c}(\cos c - \sin c) = 0$

i.e. $\cos c - \sin c = 0$

i.e. $c = \frac{\pi}{4}$

Value of $c = \frac{\pi}{4} \epsilon (0, \pi)$

Thus, Rolle's theorem is satisfied.

Question: 18

Solution:

Condition (1):

Since, $f(x)=e^{-x}(\sin x-\cos x)$ is a combination of exponential and trigonometric function which is continuous.

$\Rightarrow f(x)= e^{-x}(\sin x-\cos x)$ is continuous on $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

Condition (2):

Here, $f'(x)= e^{-x}(\sin x + \cos x) - e^{-x}(\sin x - \cos x)$

$= e^{-x}\cos x$ which exist in $[\frac{\pi}{4}, \frac{5\pi}{4}]$.

So, $f(x)= e^{-x}(\sin x-\cos x)$ is differentiable on $(\frac{\pi}{4}, \frac{5\pi}{4})$

Condition (3):

Here, $f(\frac{\pi}{4}) = e^{-\frac{\pi}{4}}(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}) = 0$

And $f(\frac{5\pi}{4}) = e^{-\frac{5\pi}{4}}(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}) = 0$

i.e. $f(\frac{\pi}{4}) = f(\frac{5\pi}{4})$

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(\frac{\pi}{4},\frac{5\pi}{4})$ such that f'(c)=0

i.e. $e^{-c}\cos c = 0$

i.e. cos c = 0

i.e. $c = \frac{\pi}{2}$

Value of $c = \frac{\pi}{2}\epsilon(\frac{\pi}{4},\frac{5\pi}{4})$

Thus, Rolle's theorem is satisfied.

Question: 19

Solution:

Condition (1):

Since, f(x) = sinx-sin2x is a trigonometric function and we know every trigonometric function is continuous.

⇒ f(x) = sinx-sin2x is continuous on [0,2π].

Condition (2):

Here, f'(x)= cosx-2cos2x which exist in [0,2π].

So, f(x)= sinx-sin2x is differentiable on (0,2π)

Condition (3):

Here, f(0)= sin0-sin0 = 0

And f(2π)=sin(2π)-sin(4π) =0

i.e. f(0)=f(2π)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one cϵ(0,2π) such that f'(c)=0

i.e. cosx-2cos2x =0

i.e. $\cos x - 4\cos^2 x + 2 = 0$

i.e. $4\cos^2 x - \cos x - 2 = 0$

i.e. $\cos x = \frac{1\pm\sqrt{33}}{8}$

i.e. c=32° 32' or c=126°23'

Value of c=32°32'ϵ(0,2π)

Thus, Rolle's theorem is satisfied.

Question: 20

Solution:

Condition (1):

Since, $f(x)=x(x+2)e^x$ is a combination of exponential and polynomial function which is continuous for all xϵR.

⇒ $f(x)= x(x+2)e^x$ is continuous on [-2,0].

Condition (2):

Here, $f'(x)=(x^2+4x+2)e^x$ which exist in [-2,0].

So, $f(x)=x(x+2)e^x$ is differentiable on (-2,0).

Condition (3):

Here, $f(-2)= (-2)(-2+2)e^{-2} =0$

And $f(0)= 0(0+2)e^0=0$

i.e. f(-2)=f(0)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(-2,0)$ such that $f'(c)=0$

i.e. $(c^2+4c+2)e^c =0$

i.e. $(c+\sqrt{2})^2=0$

i.e. $c=-\sqrt{2}$

Value of $c=-\sqrt{2}\ \epsilon(-2,0)$

Thus, Rolle's theorem is satisfied.

Question: 21

Solution:

Condition (1):

Since, $f(x)=x(x-5)^2$ is a polynomial and we know every polynomial function is continuous for all $x\epsilon R$.

$\Rightarrow f(x)= x(x-5)^2$ is continuous on [0,5].

Condition (2):

Here, $f'(x)= (x-5)^2+ 2x(x-5)$ which exist in [0,5].

So, $f(x)= x(x-5)^2$ is differentiable on (0,5).

Condition (3):

Here, $f(0)= 0(0-5)^2=0$

And $f(5)= 5(5-5)^2=0$

i.e. f(0)=f(5)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one $c\epsilon(0,5)$ such that $f'(c)=0$

i.e. $(c-5)^2+ 2c(c-5)=0$

i.e. $(c-5)(3c-5)=0$

i.e. $c = \frac{5}{3}$ or $c=5$

Value of $c = \frac{5}{3}\epsilon(0,5)$

Thus, Rolle's theorem is satisfied.

Question: 22

Solution:

Condition (1):

Since, f(x)=(x-1)(2x-3) is a polynomial and we know every polynomial function is continuous for all xϵR.

⇒ f(x)= (x-1)(2x-3) is continuous on [1,3].

Condition (2):

Here, f'(x)= (2x-3)+ 2(x-1) which exist in [1,3].

So, f(x) = (x-1)(2x-3) is differentiable on (1,3).

Condition (3):

Here, f(1)= (1-1)(2(1)-3)=0

And f(5)= (3-1)(2(3)-3)=6

i.e. f(1)≠f(3)

Condition (3) of Rolle's theorem is not satisfied.

So, Rolle's theorem is not applicable.

Question: 23

Discuss the appli

Solution:

Condition (1):

Since, f(x)=$x^{1/2}$ is a polynomial and we know every polynomial function is continuous for all xϵR.

⇒ f(x)= $x^{1/2}$ is continuous on [-1,1].

Condition (2):

Here, $f'(x) = \frac{1}{2x^{\frac{1}{2}}}$ which does not exist at x=0 in [-1,1].

f(x)=$x^{1/2}$ is not differentiable on (-1,1).

Condition (2) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

Question: 24

Solution:

Condition (1):

Since, f(x)=2+$(x-1)^{2/3}$ is a polynomial and we know every polynomial function is continuous for all xϵR.

⇒ f(x)= 2+$(x-1)^{2/3}$ is continuous on [0,2].

Condition (2):

Here, $f'(x) = \frac{2}{3(x-1)^{\frac{1}{3}}}$ which does not exist at x=1 in [0,2].

f(x)= 2+$(x-1)^{2/3}$ is not differentiable on (0,2).

Condition (2) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

Question: 25

Discuss the appli

Solution:

Condition (1):

Since, $f(x) = \cos\frac{1}{x}$ which is discontinuous at x=0

⇒ $f(x) = \cos\frac{1}{x}$ is not continuous on [-1,1].

Condition (1) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

Question: 26

Discuss the appli

Solution:

Condition (1):

Since, f(x)=[x] which is discontinuous at x=0

⇒ f(x)=[x] is not continuous on [-1,1].

Condition (1) of Rolle's theorem is not satisfied.

So,Rolle's theorem is not applicable.

Question: 27

Solution:

Condition (1):

Since, y=x(x-4) is a polynomial and we know every polynomial function is continuous for all x∈R.

⇒ y= x(x-4) is continuous on [0,4].

Condition (2):

Here, y'= (x-4)+x which exist in [0,4].

So, y= x(x-4) is differentiable on (0,4).

Condition (3):

Here, y(0)=0(0-4)=0

And y(4)= 4(4-4)=0

i.e. y(0)=y(4)

Conditions of Rolle's theorem are satisfied.

Hence, there exist at least one c∈(0,4) such that y'(c)=0

i.e. (c-4)+c=0

i.e. 2c-4=0

i.e. c=2

Value of c=2∈(0,4)

So,y(c)=y(2)=2(2-4)=-4

By geometric interpretation, (2,-4) is a point on a curve y=x(x-4),where tangent is parallel to x-axis.

Exercise : 11D

Question: 1

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(36 + 12 + 3) - (16 + 8 + 3)}{6 - 4}$$

$$= \frac{24}{2}$$

=12

⇒ f' (c)=2c+2

⇒ 2c+2=12

⇒ c=5

Question: 2

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [0,4].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(16 + 4 - 1) - (0 + 0 - 1)}{4 - 0}$$

=5

⇒ f'(c)=2c+1

⇒ 2c+1=5

⇒ c=2

Question: 3

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [1,3].

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{(18 - 9 + 1) - (2 - 3 + 1)}{3 - 1}$$

=5

⇒ f'(c)=4c-3

⇒ 4c-3=5

⇒ c=2

Question: 4

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [-1,4].

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{(64+16-24)-(-1+1+6)}{4+1}$$

$$=\frac{50}{5}$$

=10

f' (c)=3c²+2c-6

⇒ $3c^2+2c-6=10$

⇒ $3c^2+2c-16=0$

⇒ $3c^2-6c+8c-16=0$

⇒ 3c(c-2)+8(c-2)=0

⇒ (3c+8)(c-2)=0

$$c=2,\frac{-8}{3}$$

Question: 5

Solution:

Given:

$$\mathbf{f(x)=x^3-18x^2+104x-192}$$

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow f'(c)=\frac{(216-648+624-192)-(64-288+416-192)}{6-2}$$

=0

⇒ f' (c)=$3c^2-36c+104$

=$3c^2-36c+10$

=0

$$\Rightarrow c=\frac{36\pm\sqrt{1296-1248}}{6}$$

$$\Rightarrow c=\frac{36\pm\sqrt{48}}{6}$$

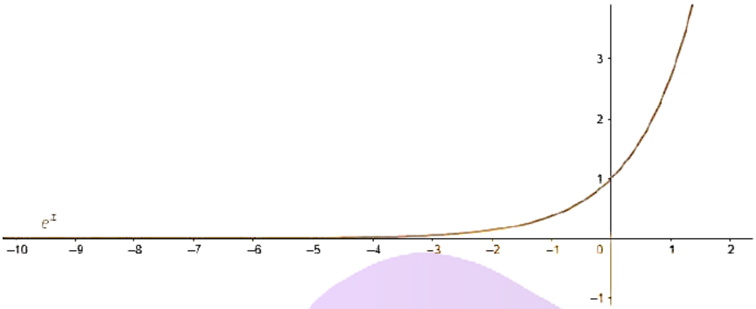
$\Rightarrow\ c = 6 \pm \frac{2}{3}\sqrt{3}$

Question: 6

Solution:

Given:

Since f(c) is continuous as well as differentiable in the interval [0,1].



$f'(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{e - 1}{1}$

$\Rightarrow f'(c) = e^c$

$\Rightarrow e^c = e - 1$

$\Rightarrow \log_e e^c = \log_e(e - 1)$

$\Rightarrow c = \log_e(e-1)$

Question: 7

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [0,1].

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{1 - 0}{1 - 0}$

$=1$

$f'(c) = \frac{2}{3}c^{\frac{1}{3}}$

$\Rightarrow \frac{2}{3}c^{\frac{1}{3}} = 1$

$\Rightarrow c^{\frac{-1}{3}} = \frac{3}{2}$

$\Rightarrow c^{\frac{1}{3}} = \frac{2}{3}$

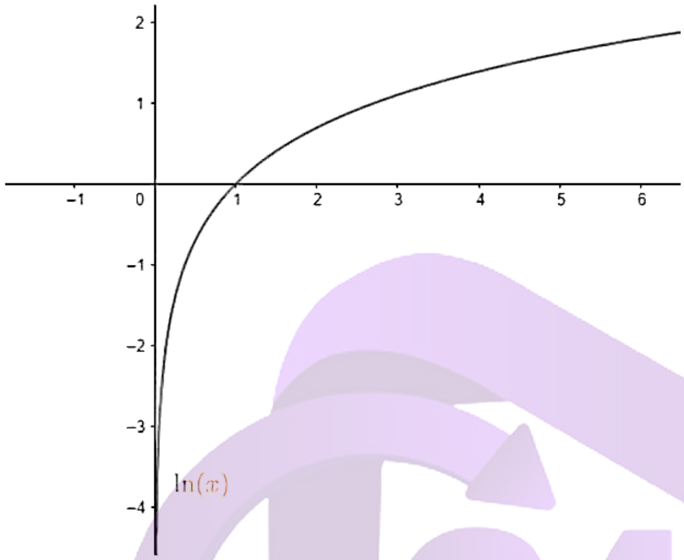
$\Rightarrow c = \frac{8}{27}$

Question: 8

Solution:

Given:

Since log x is a continuous as well as differentiable function in the interval [1,e].



$f'(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{\log e - \log 1}{e - 1}$

$= \frac{1}{e - 1}$

$f'(c) = \frac{1}{c}$

$\Rightarrow \frac{1}{e - 1} = \frac{1}{c}$

c=e-1

Question: 9

Solution:

Given:

Since $\tan^{-1} x$ is a continuous as well as differentiable function in the interval [0,1].

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{\tan^{-1} 1 - \tan^{-1} 0}{1 - 0}$

$= \frac{\pi}{4}$

$\mathbf{f'(c)} = \frac{1}{1 + c^2}$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4}$$

$$\Rightarrow 1 + c^2 = \frac{4}{\pi}$$

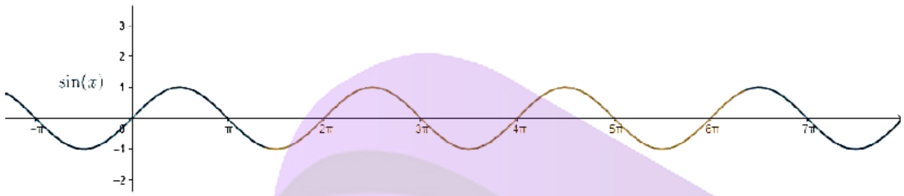
$$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1}$$

Question: 10

Solution:

Given:

Since sin x is a continuous as well as differentiable function in the interval $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin\frac{5\pi}{2} - \sin\frac{\pi}{2}}{\frac{5\pi}{2} - \frac{\pi}{2}}$$

=0

f' (c)=cos x

cos x=0

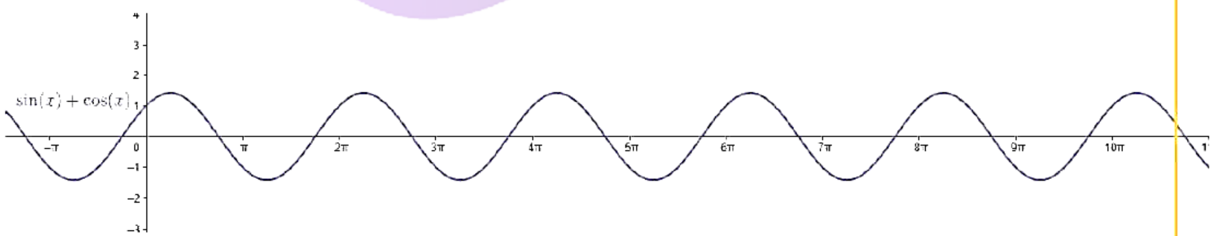
$$x = \frac{n\pi}{2}, n\epsilon\{1,2,3,4,5\}$$

Question: 11

Solution:

Given:

Since (sin x + cos x) is a continuous as well as differentiable function in the interval $\left[0, \frac{\pi}{2}\right]$.



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - \sin 0 - \cos 0}{\frac{\pi}{2} - 0}$$

=0

f' (c)=cos x-sin x

$\Rightarrow$ cos x-sin x=0

$$\Rightarrow \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4} = 0$$

$$\Rightarrow \cos\left(x + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \left(x + \frac{\pi}{4}\right) = \cos^{-1} 0$$

$$\Rightarrow \left(x + \frac{\pi}{4}\right) = 1$$

$$\Rightarrow \mathbf{x = 1 - \frac{\pi}{4}}$$

Question: 12

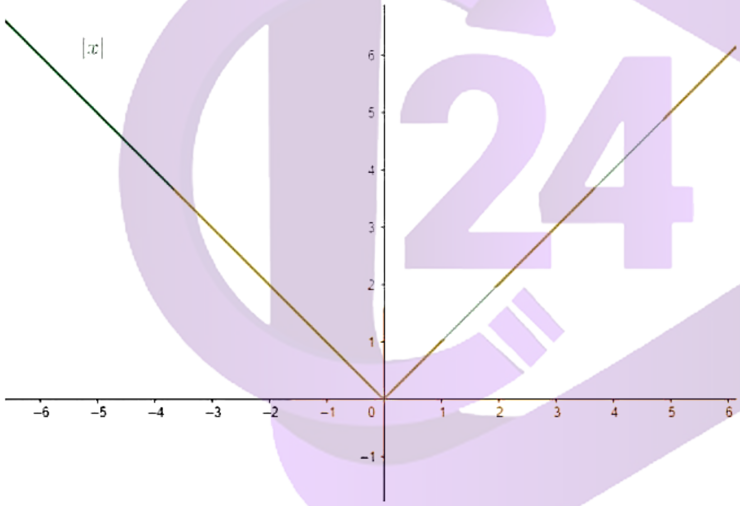
Solution:

Given:

Since f(x) is continuous in the interval [-1,1].

But is non differentiable at x=0 due to sharp corner.

So LMVT is not applicable to $f(x)=|x|$

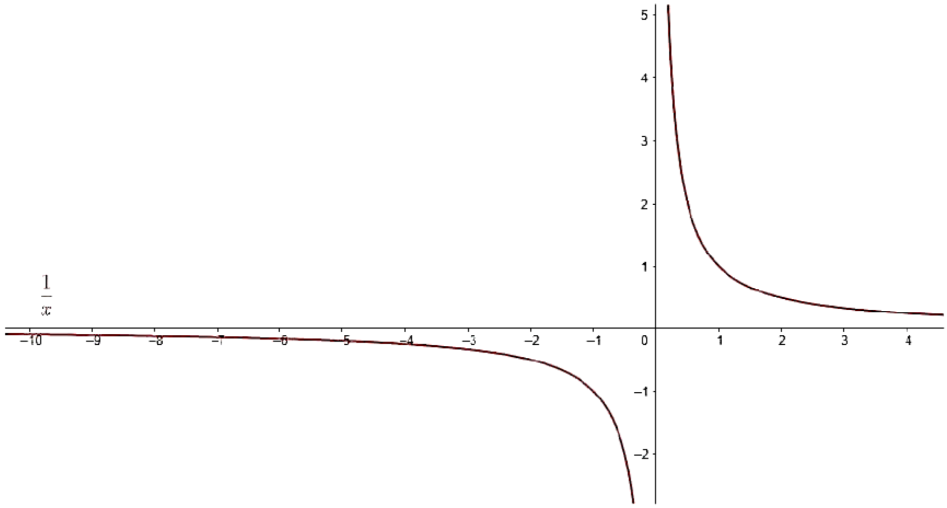


Question: 13

Solution:

Given:

Since the graph is discontinuous at x=0 as shown in the graph.



So LMVT is not applicable to the above function.

Question: 14 A

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval $[0,\frac{1}{2}]$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{\frac{1}{8} - \frac{3}{4} + 1 - 0}{\frac{1}{2} - 0}$$

$$= \frac{3}{4}$$

f' $(c)=3x^2 -6x+2$

$3 x^2-6x+2=3/4$

$12 x^2-24x+8=3$

$12 x^2-24x+5=0$

$$x = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$x = 1 \pm \sqrt{\frac{336}{576}}$$

$$x = 1 \pm \sqrt{\frac{7}{12}}$$

Question: 14 B

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [1,5].

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{\sqrt{25-25}-\sqrt{25-1}}{5-1}$$

$$=\frac{-\sqrt{24}}{4}$$

$$f'(c)=\frac{1}{2\sqrt{25-c^2}}(-2c)$$

$$\Rightarrow\ \frac{-c}{\sqrt{25-c^2}}=\frac{-\sqrt{24}}{4}$$

$$\Rightarrow\ 4c=\sqrt{24(25-c^2)}$$

$\Rightarrow\ 16c^2=600-24c^2$

$\Rightarrow\ 40c^2=600$

$\Rightarrow\ c^2=15$

$$\Rightarrow\ c=\sqrt{15}$$

Question: 14 C

Solution:

Given:

Since the f(x) is a polynomial function,

It is continuous as well as differentiable in the interval [4,6].

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{\sqrt{8}-\sqrt{6}}{6-4}$$

$$=\frac{\sqrt{8}-\sqrt{6}}{2}$$

$$\mathbf{f'(c)}=\frac{1}{2\sqrt{c+2}}$$

$$\Rightarrow\ \frac{1}{2\sqrt{c+2}}=\frac{\sqrt{8}-\sqrt{6}}{2}$$

$$\Rightarrow\ \frac{1}{\sqrt{c+2}}=\frac{\sqrt{8}-\sqrt{6}}{1}$$

$$\Rightarrow\ \sqrt{c+2}=\frac{1}{\sqrt{8}-\sqrt{6}}\times\frac{\sqrt{8}+\sqrt{6}}{\sqrt{8}+\sqrt{6}}$$

$$\Rightarrow\ \sqrt{c+2}=\frac{\sqrt{8}+\sqrt{6}}{2}$$

$$\Rightarrow\ \mathbf{c+2}=\frac{1}{4}\left(8+6+2\sqrt{48}\right)$$

$$\Rightarrow\ \mathbf{c}=\frac{\mathbf{3}}{\mathbf{2}}+\mathbf{2\sqrt{3}}$$

⇒ c=4.964

Question: 15

Solution:

Given:

$y=x^2$

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{4-1}{2-1}$$

$$=3$$

⇒ $f'(c)=2c$

⇒ $2c=3$

$$c=\frac{3}{2}$$

So, the point is $\left(\frac{3}{2},\frac{9}{4}\right)$

Question: 16

Solution:

Given:

$y=x^3$

Since y is a polynomial function.

It is continuous and differentiable in [1,3]

So, there exists a c such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{27-1}{3-1}$$

$$=13$$

⇒ $f'(c)=3c^2$

⇒ $3c^2=13$

$$\Rightarrow c=\sqrt{\frac{13}{3}}$$

$$\Rightarrow c=\frac{\sqrt{39}}{3}$$

So the point is $\left(\frac{\sqrt{39}}{3},\frac{13\sqrt{39}}{9}\right)$

Question: 17

Solution:

Given:

$y=x^3-3x$

Since y is a polynomial function.

It is continuous and differentiable in [1,2]

So, there exists a c such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$=\frac{(8-6)-(1-3)}{2-1}$$

=4

$\Rightarrow f'(c)=3c^2-3$

$\Rightarrow 3c^2-3=4$

$\Rightarrow 3c^2=7$

$$\Rightarrow c^2=\frac{7}{3}$$

$$\Rightarrow c=\pm\sqrt{\frac{7}{3}}$$

So, the points are $\left(\sqrt{\frac{7}{3}},\frac{-2}{3}\sqrt{\frac{7}{3}}\right),\left(\frac{-7}{3}\sqrt{\frac{7}{3}},\frac{2}{3}\sqrt{\frac{7}{3}}\right)$

Question: 18

Solution:

Given:

$f(x)=x(1-\log x)$

Since the function is continuous as well as differentiable

So, there exists c such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow (1-\log c)-c\times\frac{1}{c}=\frac{b(1-\log b)-a(1-\log a)}{b-a}$$

$$\Rightarrow \log c=\frac{b(1-\log b)-a(1-\log a)}{b-a}$$

$(b-a)\log c=b(1-\log b)-a(1-\log a)$

Hence proved.

Exercise : 11E

Question: 1

Solution:

min. value = 4

Since the square of any no. Is positive, the given function has no maximum value.

The minimum value exists when the quantity $(5x-1)^2=0$

Therefore, minimum value=4

Question: 2

Solution:

max. value = 9

Since the quantity $(x-3)^2$ has a –ve sign, the max. Value it can have is 9.

Also hence it has no minimum value.

Question: 3

Find the maximum

Solution:

max. value = 6

Since $|x+4|$ is non-negative for all x belonging to R.

Therefore the least value it can have is 0 .

Hence value of function is 6.

It has no minimumvalue as it can have infinitely many.

Question: 4

Solution:

max. value = 4, min. value = 6

$f(x)=\sin 2x+5$

We know that,

$-1\le\sin\theta\le1$

$-1\le\sin 2x\le1$

Adding 5 on both sides,

$-1+5\le\sin 2x+5\le1+5$

$4\le\sin 2x+5\le6$

Hence

max value of $f(x)=2x+5$ will be 6

Min value of $f(x)=2x+5$ will be 4

Question: 5

Solution:

max. value = 4, min. value = 2

We know that

$-1\le\sin\theta\le1$

$-1\le\sin 4x\le1$

Adding 3 on both sides,

We get

$-1+3 \le \sin 4x+3 \le 1+3$

$2 \le |\sin 4x+3| \le 4$

Hence min.Value is 2 and max value is 4

Question: 6

Solution:

local max. value is 0 at x = 3

$F'(x)=4(x-3)^3=0$

$\Rightarrow X=3$

∴ local max. Vaue is 0.

Question: 7

Solution:

local min. value is 0 at x = 0

$F'(x)=2x=0$

$x=0$

∴ local min.value is 0

Question: 8

Solution:

local max. value is −3 at x = 1 and local min. value is −128 at x = 6

$F'(x)=6x^2-42x+36=0$

$\Rightarrow 6(x-1)(x-6)=0$

$\Rightarrow x=1,6$

$F''(x)=12x-42$

$F''(1)<0$,1 is the pont of local max.

$F''(6)>0$, 6 is the point of localmin.

$F(1)=2-21+36-20=-3$

$F(6)=-128$

Question: 9

Solution:

local max. value is 19 at x = 1 and local min. value is 15 at x = 3

$F'(x)=3x^2-12x+9=0$

$\Rightarrow 3(x-3)(x-1)=0$

$\Rightarrow x=3,1$

$F''(x)=6x-12$

$F''(3)=18-12=6>0$, 3 is the of local min.

$F''(1)<0$, 1 is the point of local max.

F(3)=15

F(1)=19

Question: 10

Solution:

local max. value is 68 at x = 1 and local min. values are −1647 at x = −6 and −316 at x = 5

$F'(x)=4x^3-124x+120=0$

$\Rightarrow 4(x^3-31x+30)=0$

For x=1, the given eq is 0

∴x-1 is a factor,

$4(x-1)(x+6)(x-5)=0$

$\Rightarrow X=1,-6,5$

$F''(1)<0$, 1is the point of max.

$F''(-6)$ and $f''(5)>0$, -6 and 5 are point of min.

F(1)=68

F(-6)=-1647

F(5)=-316

Question: 11

Solution:

local max. value is 251 at x = 8 and local min. value is −5 at x = 0

$f'(x)=-3x^2+24x=0$

$\Rightarrow -3x(x-8)=0$

$\Rightarrow x=0,8$

$F''(x)=-6x+24$

$F''(0)>0$, 0 is the point of local min.

$F''(8)<0$, 8 is the point of local max.

F(8)=251 and f(0)=-5

Question: 12

Solution:

local max. value is 0 at x = −2 and local min. value is −4 at x = 0

$f'(x)=(x-1)2(x+2)+(x+2)^2=0$

x=0,-2

$f''(0)>0$, 0 is the point of local min.

$f''(-2)<0$, -2 is the point of local max.

$f(0)=-4$

$f(-2)=0$

Question: 13

Solution:

local max. value is 0 at each of the points x = 1 and x = −1 and local min. value is $\frac{-3456}{3125}$ at $x=-\frac{1}{5}$

$F'(x)=-(x-1)^3 2(x+1)-3(x-1)^2(x+1)^2=0$

$\Rightarrow x = 1, -1, -\frac{1}{5}$

Since, $f''(1)$ and $f''(-1) < 0$, 1 and -1 are the points of local max.

$F''(-\frac{1}{5})>0$, $-\frac{1}{5}$ is the point of local min.

$F(1)=f(-1)=0$

Also, $f\left(-\frac{1}{5}\right) = -\frac{3456}{3125}$

Question: 14

Solution:

local min. value is 2 at x = 2

$F'(x)=\frac{1}{2}-\frac{2}{x^2}=0$

$\Rightarrow x^2-4=0$

$\Rightarrow x=\pm 2$

But since $x>0$, $x=2$

$F''(2)=\frac{2}{x^3}$

$=\frac{2}{8}<0$

∴point of local mini. is 2

$F(2)=\frac{2}{2}+\frac{2}{2}=2$

Question: 15

Solution:

max. value is 139 at x = −2 and min. value is 89 at x = 3

$F'(x)=6x^2-24=0$

$6(x^2-4)=0$

$6(x^2-2^2)=0$

$6(x-2)(x+2)=0$

X=2,-2

Now, we shall evaluate the value of f at these points and the end points

$F(2)=2(2)^3-24(2)+107=75$

$F(-2)=2(-2)^3-24(-2)+107=139$

$F(-3)=2(-3)^3-24(-3)+107=125$

$F(3)=2(3)^3-24(3)+107=89$

Question: 16

Solution:

max. value is 257 at x = 4 and min. value is −63 at x = 2

$F^{I}(x)=12x^3-24x^2+24x-48=0$

$12(x^3-2x^2+2x-4)=0$

Since for x=2, $x^3-2x^2+2x-4=0$, x-2 is a factor

On dividing x^3-2x^2+2x-4 by x-2, we get,

$12(x-2)(x^2+2)=0$

X=2,4

Now, we shall evaluate the value of f at these points and the end points

$F(1)=3(1)^4-8(1)^3+12(1)^2-48(1)+1=-40$

$F(2)= 3(2)^4-8(2)^3+12(2)^2-48(2)+1=-63$

$F(4)= 3(4)^4-8(4)^3+12(4)^2-48(4)+1=257$

Question: 17

Solution:

max. value is $\frac{3}{4}$ at $x=\frac{\pi}{6}$ and min. value is $\frac{1}{2}$ at $x=\frac{\pi}{2}$

$F^{I}(x)=\cos x-\frac{1}{2}\sin x=0$

ð 2 cos x=sin x

$\Rightarrow \frac{\pi}{6}=\frac{\pi}{3}$

$f\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}+\frac{1}{2}\cos\frac{\pi}{2}=\frac{1}{2}$

$f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}+\frac{1}{2}\cos\frac{\pi}{6}=\frac{1}{2}+\frac{\sqrt{3}}{4}$

$f\left(\frac{\pi}{3}\right)=\sin\frac{\pi}{3}+\frac{1}{2}\cos\frac{\pi}{3}=\frac{\sqrt{3}}{2}+\frac{1}{4}$

Question: 18

Solution:

The given function is

$Y = x^{\frac{1}{x}}$

Now, taking logarithm from both sides, we get..

$$\log y = \frac{1}{x}\log x$$

Differentiating both sides w.r.t x....

$$\frac{1}{y}y' = -\frac{1}{x^2}\ln(x) + \frac{1}{x^2}$$

$$\Rightarrow y' = \frac{y}{x^2}(1 - \ln(x))$$

(1-ln(x))=0

ln(x)=1

x=e

hence the max. point is x=e

max value is $e^{\frac{1}{e}}$.

Question: 19

Solution:

F(x)=x+ $\frac{1}{x}$

Taking first derivative and equating it to zero to find extreme points.

$F'(x)=1-\frac{1}{x^2}=0$

$X^2=1$

x=1,x=-1

now to determine which of these is min. And max. We use second derivative.

$f''(x)=\frac{2}{x^3}$

$f''(1)=2$ and $f''(-1)=-2$

since $f''(1)$ is +ve it is minimum point while $f''(-1)$ is –ve it is maximum point

max value-> $f(-1)=-1+\frac{1}{-1}=-2$

min vaue-> $f(1)=1+\frac{1}{1}=2$

hence maximum value is less than minimum value

Question: 20

Solution:

49

$$\frac{dp}{dx} = -24 - 36x$$

=0

$\Rightarrow x=-23$

Step 2

$\frac{d^2p}{dx^2}=-36$ is negative

Step 3

maximum profit $= p\left(-\frac{2}{3}\right)$

$=49$

Question: 21

Solution:

(1, 3)

Let P(x,y) be the position of the jet and the soldier is placed at A(3,2)

AP= $\sqrt{(x-3)^2+(y-2)^2}$

As $y=x^2+2$ or $y-2=x^2$

$\therefore AP^2=(x-3)^2+x^4=z$ (say)

$$\frac{dz}{dx}=2(x-3)+4x^3$$

$$\frac{dz}{dx}=0$$

$2x-6+4x^3=0$

Put x=1

2-6+4=0

$\therefore$ x-1 is a factor

And $\frac{d^2z}{dx^2}=12x^2+2$

$\frac{dz}{dx}=0$ or x=1

and $\frac{d^2z}{dx^2}$(at x=1)>0

$\therefore$ z is minimum when x=1,y=1+2=3

Point is (1,3)

Question: 22

Solution:

max. value is $\left(-\frac{\pi}{3}+\sqrt{3}\right)$ at $x=\frac{\pi}{3}$ and min. value is $\left(\frac{5\pi}{3}+\sqrt{3}\right)$ at $x=\frac{5\pi}{3}$

$f'(x)=-1+2\cos x=0$

$$\Rightarrow \cos x=-\frac{1}{2}$$

$$x=\frac{2\pi}{3}$$

By finding the general solution, we get $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$

Now, by finding the second derivative, we get that $f''(\frac{\pi}{3}) < 0$ and $f''(\frac{5\pi}{3}) > 0$

Therefore, max. value is $\left(-\frac{\pi}{3} + \sqrt{3}\right)$ at $x = \frac{\pi}{3}$ and min. value is $\left(\frac{5\pi}{3} + \sqrt{3}\right)$ at $x = \frac{5\pi}{3}$

Exercise : 11F

Question: 1

Solution:

Given,

- The two numbers are positive.
- the product of two numbers is 49.
- the sum of the two numbers is minimum.

Let us consider,

- x and y are the two numbers, such that $x > 0$ and $y > 0$
- Product of the numbers : $x \times y = 49$
- Sum of the numbers : $S = x + y$

Now as,

$x \times y = 49$

$y = \frac{49}{x}$ ------- (1)

Consider,

$S = x + y$

By substituting (1), we have

$S = x + \frac{49}{x}$ ------- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}\left(x + \frac{49}{x}\right)$$

$$\frac{dS}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{49}{x}\right)$$

$$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right) \text{ ----- (3)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 1 + 49\left(\frac{-1}{x^2}\right) = 0$$

$$= 1 - \left(\frac{49}{x^2}\right) = 0$$

$= 1 = \left(\frac{49}{x^2}\right)$

$= x^2 = 49$

$= x = \pm\sqrt{49}$

As x > 0, then x = 7

Now, for checking if the value of S is maximum or minimum at x=7, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 7.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[1 + 49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[1] + \frac{d}{dx}\left[49\left(\frac{-1}{x^2}\right)\right]$$

$$\frac{d^2S}{dx^2} = 0 + \left[49\left(\frac{-1 \times -2}{x^3}\right)\right]$$

[Since $\frac{d}{dx}$ (constant) = 0 and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2S}{dx^2} = 49\left(\frac{2}{x^3}\right) = \frac{98}{x^3}$$

Now when x = 7,

$$\left[\frac{d^2S}{dx^2}\right]_{x=7} = \frac{98}{7^3} = \frac{98}{343} > 0$$

As second differential is positive, hence the critical point x = 7 will be the minimum point of the function S.

Therefore, the function S = sum of the two numbers is minimum at x = 7.

From Equation (1), if x= 7

$$y = \frac{49}{7} = 7$$

Therefore, x = 7 and y = 7 are the two positive numbers whose product is 49 and the sum is minimum.

Question: 2

Solution:

Given,

- The two numbers are positive.
- the sum of two numbers is 16.
- the sum of the squares of two numbers is minimum.

Let us consider,

- x and y are the two numbers, such that x > 0 and y > 0
- Sum of the numbers : x + y = 16
- Sum of squares of the numbers : $S = x^2 + y^2$

Now as,

x + y = 16

y = (16-x) ------(1)

Consider,

$S = x^2 + y^2$

By substituting (1), we have

$S = x^2 + (16-y)^2$ --------(2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}\left[x^2 + (16-x)^2\right]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}\left[(16-x)^2\right]$$

$$\frac{dS}{dx} = 2x + 2(16-x)(-1) \text{ ----- (3)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 2x + 2(16-x)(-1) = 0$$

$\Rightarrow 2x - 2(16 - x) = 0$

$\Rightarrow 2x - 32 + 2x = 0$

$= 4x = 32$

$$\Rightarrow x = \frac{32}{4}$$

$\Rightarrow x = 8$

As x > 0, x = 8

Now, for checking if the value of S is maximum or minimum at x=8, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 8.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[2x + 2(16-x)(-1)\right]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[2x] - 2\frac{d}{dx}[16-x]$$

$$\frac{d^2S}{dx^2} = 2 - 2[0-1]$$

[Since $\frac{d}{dx}(\text{constant}) = 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{d^2S}{dx^2} = 2 - 0 + 2 = 4$$

Now when x = 8,

$$\left[\frac{d^2S}{dx^2}\right]_{x=8} = 4 > 0$$

As second differential is positive, hence the critical point x = 8 will be the minimum point of the function S.

Therefore, the function S = sum of the squares of the two numbers is minimum at x = 8.

From Equation (1), if x= 8

y = 16 – 8 = 8

Therefore, x = 8 and y = 8 are the two positive numbers whose su is 16 and the sum of the squares is minimum.

Question: 3

Solution:

Given,

- the number 15 is divided into two numbers.
- the product of the square of one number and cube of another number is maximum.

Let us consider,

- x and y are the two numbers
- Sum of the numbers : x + y = 15
- Product of square of the one number and cube of anther number : $P = x^3 y^2$

Now as,

x + y = 15

y = (15-x) ------(1)

Consider,

$P = x^3y^2$

By substituting (1), we have

$P = x^3 \times (15-x)^2$ --------(2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}[x^3 \times (15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2\frac{d}{dx}(x^3) + x^3\frac{d}{dx}[(15-x)^2]$$

$$\frac{dP}{dx} = (15-x)^2(3x^2) + x^3[2(15-x)(-1)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = (15-x)^2(3x^2) + x^3[-30 + 2x]$$

$= 3\times[15^2 - 2\times(15)\times(x) + x^2]x^2 + x^3(2x-30)$

$= x^2[3\times(225 - 30x + x^2) + x(2x - 30)]$

$= x^2[675 - 90x + 3x^2 + 2x^2 - 60x]$

$= x^2[5x^2 - 120x + 675]$

$= 5x^2 [x^2 - 24x + 135]$ ----- (3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = 5x^2[x^2 - 24x + 135] = 0$$

Hence $5x^2 = 0$ (or) $x^2 - 24x + 135 = 0$

$x = 0$ (or) $x = \frac{-(-24)\pm\sqrt{(-24)^2-4(1)(135)}}{2\times1}$

$x = 0$ (or) $x = \frac{24\pm\sqrt{576-540)}}{2}$

$x = 0$ (or) $x = \frac{24\pm\sqrt{36}}{2}$

$x = 0$ (or) $x = \frac{24\pm6}{2}$

$x = 0$ (or) $x = \frac{24+6}{2}$ (or) $x = \frac{24-6}{2}$

$x = 0$ (or) $x = \frac{30}{2}$ (or) $x = \frac{18}{2}$

$x = 0$ (or) $x = 15$ (or) $x = 9$

Now considering the critical values of $x = 0,9,15$

Now, for checking if the value of P is maximum or minimum at x=0,9,15, we will perform the second differentiation and check the value of $\frac{d^2P}{dx^2}$ at the critical value $x = 0,9,15$.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2P}{dx^2} = \frac{d}{dx}[5x^2(x^2 - 24x + 135)]$$

$$\frac{d^2P}{dx^2} = (x^2 - 24x + 135)\frac{d}{dx}[5x^2] + 5x^2\frac{d}{dx}[x^2 - 24x + 135]$$

$= (x^2 - 24x + 135)(5 \times 2x) + 5x^2(2x - 24 + 0)$

[Since $\frac{d}{dx}$(constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$= (x^2 - 24x + 135)(10x) + 5x^2(2x - 24)$

$= 10x^3 - 240x^2 + 1350x + 10x^3 - 120x^2$

$= 20x^3 - 360x^2 + 1350x$

$= 5x(4x^2 - 72x + 270)$

$$\frac{d^2P}{dx^2} = 5x(4x^2 - 72x + 270)$$

Now when $x = 0$,

$$\left[\frac{d^2P}{dx^2}\right]_{x=0} = 5 \times 0[4(0)^2 - 72(0) + 270]$$

$= 0$

So, we reject $x = 0$

Now when $x = 15$,

$$\left[\frac{d^2P}{dx^2}\right]_{x=15} = 5 \times 15\,[4(15)^2 - 72\,(15) + 270]$$

= 65 [(4 × 225) –1080+ 270]

= 65 [900– 1080+ 270]

= 65 [1170– 1080]

= 65× (90) > 0

Hence $\left[\frac{d^2P}{dx^2}\right]_{x=15} > 0$, so at x = 15, the function P is minimum

Now when x = 9,

$$\left[\frac{d^2P}{dx^2}\right]_{x=9} = 5 \times 9\,[4(9)^2 - 72\,(9) + 270]$$

= 45 [(4 × 81) – 648 + 270]

= 45 [324 – 648 + 270]

= 45 [594 – 648]

= 45 × (-54)

= -2430 < 0

As second differential is negative, hence at the critical point x = 9 will be the maximum point of the function P.

Therefore, the function P is maximum at x = 9.

From Equation (1), if x= 9

y = 15 – 9 = 6

Therefore, x = 9 and y = 6 are the two positive numbers whose sum is 15 and the product of the square of one number and cube of another number is maximum.

Question: 4

Solution:

Given,

- the number 8 is divided into two numbers.
- the product of the square of one number and cube of another number is minimum.

Let us consider,

- x and y are the two numbers
- Sum of the numbers : x + y = 8
- Product of square of the one number and cube of anther number : $S = x^3 + y^2$

Now as,

x + y = 8

y = (8-x) ------ (1)

Consider,

$S = x^3 + y^2$

By substituting (1), we have

$S = x^3 + (8-x)^2$---------(2)

For finding the maximum/ minimum of given function, we can find it by differentiatin and then equating it to zero. This is because if the function f(x) has a maximum/minir point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dS}{dx} = \frac{d}{dx}[x^3 + (8-x)^2]$$

$$\frac{dS}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}[(8-x)^2]$$

$$\frac{dS}{dx} = (3x^2) + 2(8-x)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dS}{dx} = 3x^2 - 16 + 2x$$

$= 3x^2 + 2x - 16$ ------(3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dS}{dx} = 3x^2 + 2x - 16 = 0$$

Hence $3x^2 + 2x - 16 = 0$

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-16)}}{2 \times 3}$$

$$= \frac{-2 \pm \sqrt{4 + 192}}{6}$$

$$= \frac{-2 \pm \sqrt{196}}{6}$$

$$x = \frac{-2 \pm 14}{6}$$

$x = \frac{-2+14}{6}$ (or) $x = \frac{-2-14}{6}$

$x = \frac{12}{6}$ (or) $x = \frac{-16}{6}$

x = 2 (or) x = -2.67

Now considering the critical values of x = 2,-2.67

Now, for checking if the value of P is maximum or minimum at x=2,-2.67, we will perform the second differentiation and check the value of $\frac{d^2S}{dx^2}$ at the critical value x = 2,-2.67.

Performing the second differentiation on the equation (3) with respect to x.

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[3x^2 + 2x - 16]$$

$$\frac{d^2S}{dx^2} = \frac{d}{dx}[3x^2] + \frac{d}{dx}[2x] - \frac{d}{dx}[16]$$

$= 3(2x) + 2(1) - 0$

[Since $\frac{d}{dx}$(constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$= 6x + 2$

$\frac{d^2S}{dx^2} = 6x + 2$

Now when x = -2.67,

$\left[\frac{d^2S}{dx^2}\right]_{x=-2.67} = 6\,(-2.67) + 2$

= - 16.02 + 2 = -14.02

At x = -2.67 $\frac{d^2S}{dx^2} = -14.02 < 0$ hence, the function S will be maximum at this point.

Now consider x = 2,

$\left[\frac{d^2S}{dx^2}\right]_{x=2} = 6\,(2) + 2$

= 12 + 2 = 14

Hence $\left[\frac{d^2S}{dx^2}\right]_{x=2} = 14 > 0$, so at x = 2, the function S is minimum

As second differential is positive, hence at the critical point x = 2 will be the maximum point of the function S.

Therefore, the function S is maximum at x = 2.

From Equation (1), if x= 2

y = 8 – 2 = 6

Therefore, x = 2 and y = 6 are the two positive numbers whose sum is 8 and the sum of the square of one number and cube of another number is maximum.

Question: 5

Solution:

Given,

- the number 'a' is divided into two numbers.
- the product of the pth power of one number and qth power of another number is maximum.

Let us consider,

- x and y are the two numbers
- Sum of the numbers : x + y = a
- Product of square of the one number and cube of anther number : $P = x^p y^q$

Now as,

x + y = a

y = (a-x) ------ (1)

Consider,

$P = x^p y^q$

By substituting (1), we have

$P = x^p \times (a-x)^q$ -------- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with x

$$\frac{dP}{dx} = \frac{d}{dx}[x^p \times (a-x)^q]$$

$$\frac{dP}{dx} = (a-x)^q \frac{d}{dx}(x^p) + x^p \frac{d}{dx}[(a-x)^q]$$

$$\frac{dP}{dx} = (a-x)^q (px^{p-1}) + x^p [q(a-x)^{q-1}(-1)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u \times v) = v \times \frac{d}{dx}(u) + u \times \frac{d}{dx}(v)$]

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[(a-x)p - xq]$$

$= x^{p-1}(a-x)^{q-1}[ap-xp-xq]$

$= x^{p-1}(a-x)^{q-1}[ap - x(p+q)]$ -----(3)

Now equating the first derivative to zero will give the critical point c.

So,

$$\frac{dP}{dx} = x^{p-1}(a-x)^{q-1}[ap - x(p+q)] = 0$$

Hence $x^{p-1} = 0$ (or) $(a-x)^{q-1}$ (or) $ap - x(p+q) = 0$

$x = 0$ (or) $x = a$ (or) $x = \frac{ap}{p+q}$

Now considering the critical values of $x = 0, a$ and $x = \frac{ap}{p+q}$

Now, using the First Derivative test,

For f, a continuous function which has a critical point c, then, function has the local maximum at c, if f'(x) changes the sign from positive to negative as x increases through c, i.e. f'(x)>0 at every point close to the left of c and f'(x)<0 at every point close to the right of c.

Now when $x = 0$,

$$\left[\frac{dP}{dx}\right]_{x=0} = 0$$

So, we reject $x = 0$

Now when $x = a$,

$$\left[\frac{dP}{dx}\right]_{x=a} = 0$$

Hence we reject $x = a$

Now when $x < \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x<\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] > 0 \text{ ---- } (4)$$

Now when $x > \frac{ap}{p+q}$,

$$\left[\frac{dP}{dx}\right]_{x>\frac{ap}{p+q}} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}\left[ap - \frac{ap}{p+q}(p+q)\right] < 0 \text{ ---- } (5)$$

By using first derivative test, from (4) and (5), we can conclude that, the function P has local maximum at $x = \frac{ap}{p+q}$

From Equation (1), if $x = \frac{ap}{p+q}$

$y = a - \frac{ap}{p+q} = \frac{a(p+q) - ap}{p+q} = \frac{ap}{p+q}$

Therefore, $x = \frac{ap}{p+q}$ and $y = \frac{aq}{p+q}$ are the two positive numbers whose sum together to give the number 'a' and whose product of the pth power of one number and qth power of the other number is maximum.

Question: 6

Solution:

Given:

Rate of working of an engine R, v is the speed of the engine:

$R = 15v + \frac{6000}{v}$, where 0<v<30

For finding the maximum/ minimum of given function, we can find it by differentiating it with v and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Now, differentiating the function R with respect to v.

$\frac{dR}{dv} = \frac{d}{dv}[15v + \frac{6000}{v}]$

$\frac{dR}{dv} = \frac{d}{dv}[15v] + \frac{d}{dv}\left[\frac{6000}{v}\right]$

$\frac{dR}{dv} = 15 + \left[\frac{6000}{v^2}\right](-1) = 15 - \frac{6000}{v^2}$ ----- (1)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Equating equation (1) to zero to find the critical value.

$\frac{dR}{dv} = 15 - \frac{6000}{v^2} = 0$

$15 = \frac{6000}{v^2}$

$v^2 = \frac{6000}{15} = 400$

$v^2 = 400$

$v = \pm\sqrt{400}$

v = 20 (or) v = -20

As given in the question 0<v<30, v = 20

Now, for checking if the value of R is maximum or minimum at v=20, we will perform the second differentiation and check the value of $\frac{d^2R}{dv^2}$ at the critical value v = 20.

Differentiating Equation (1) with respect to v again:

$\frac{d^2R}{dv^2} = \frac{d}{dx}[15 - \frac{6000}{v^2}]$

$= \frac{d}{dx}[15] - \frac{d}{dx}\left[\frac{6000}{v^2}\right]$

$= 0 - (-2)\left[\frac{6000}{v^3}\right]$

[Since $\frac{d}{dx}$(constant) = 0 and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$= 2\left[\frac{6000}{v^3}\right]$$

$$\frac{d^2R}{dv^2} = \left[\frac{12000}{v^3}\right] \text{ ----- (2)}$$

Now find the value of $\left(\frac{d^2R}{dv^2}\right)_{v=20}$

$$\left(\frac{d^2R}{dv^2}\right)_{v=20} = \left[\frac{12000}{(20)^3}\right] = \frac{12000}{20 \times 20 \times 20} = \frac{3}{2} > 0$$

So, at critical point v = 20. The function R is at its minimum.

Hence, the function R is at its minimum at v = 20.

Question: 7

Solution:

Given,

- Area of the rectangle is 93 cm^2.
- The perimeter of the rectangle is also fixed.

Let us consider,

- x and y be the lengths of the base and height of the rectangle.
- Area of the rectangle = A = x × y = 96 cm^2
- Perimeter of the rectangle = P = 2 (x + y)

As,

x × y = 96

$y = \frac{96}{x}$ ....(1)

Consider the perimeter function,

P = 2 (x + y)

Now substituting (1) in P,

$P = 2\left(x + \frac{96}{x}\right)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2\left(x + \frac{96}{x}\right)\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}\left(\frac{96}{x}\right)$$

$$\frac{dP}{dx} = 2\,(1) + \ 2\left(\frac{96}{x^2}\right)\,(-1)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \ \frac{d}{dx}\,(x^{-n}) = \ -nx^{-n-1}$]

$$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right) \text{----- } (3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \left(\frac{192}{x^2}\right) = 0$$

$$2 = \left(\frac{192}{x^2}\right)$$

$$x^2 = \left(\frac{192}{2}\right) = 96$$

$$x = \ \sqrt{96}$$

$$x = \ \pm 4\,\sqrt{6}$$

As the length and breadth of a rectangle cannot be negative, hence $x = \ 4\,\sqrt{6}$

Now to check if this critical point will determine the least perimeter, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \left(\frac{192}{x^2}\right)\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \ \frac{d}{dx}\left(\frac{192}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \ (-2)\left(\frac{192}{x^3}\right)$$

[Since $\frac{d}{dx}$ (constant) = 0 and $\frac{d}{dx}\left(\frac{1}{x^n}\right) = \ \frac{d}{dx}\,(x^{-n}) = \ -nx^{-n-1}$]

$$\frac{d^2P}{dx^2} = \left(\frac{2 \times 192}{x^3}\right) \text{.........}(4)$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}}$

$$\frac{d^2P}{dx^2} = \left(\frac{2\ \times 192}{(4\sqrt{6})^3}\right)$$

$$= \left(\frac{2\ \times 192}{4\sqrt{6}\ \times 4\sqrt{6} \times\ 4\sqrt{6}}\right)$$

$$= \left(\frac{2\ \times 192}{4\sqrt{6}\ \times 4\sqrt{6} \times\ 4\sqrt{6}}\right) = \ \frac{1}{\sqrt{6}}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=4\sqrt{6}} = \frac{1}{\sqrt{6}} > 0$, so the function P is minimum at $x = 4\sqrt{6}$.

Now substituting $x = 4\sqrt{6}$ in equation (1):

$$y = \frac{96}{4\sqrt{6}}$$

$$y = \frac{96\,\sqrt{6}}{4\ \times 6}$$

[By rationalizing he numerator and denominator with $\sqrt{6}$]

$$\therefore\ y = \ 4\sqrt{6}$$

Hence, area of the rectangle with sides of a rectangle with $x = 4\sqrt{6}$ and $y = 4\sqrt{6}$ is 96 has the least perimeter.

Now the perimeter of the rectangle is

$$P = 2(4\sqrt{6} + 4\sqrt{6}) = 2(8\sqrt{6}) = 16\sqrt{6}\text{ cms}$$

The least perimeter is $16\sqrt{6}$ cms.

Question: 8

Solution:

Given,

- Rectangle with given perimeter.

Let us consider,

- 'p' as the fixed perimeter of the rectangle.
- 'x' and 'y' be the sides of the given rectangle.
- Area of the rectangle, A = x × y.

Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the area of the rectangle,

A = x × y

Substituting (1) in the area of the rectangle,

$$A = x \times \left(\frac{p-2x}{2}\right)$$

$A = \frac{1}{2} \times (px - 2x^2)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{2}(px - 2x^2)\right]$$

$$\frac{dA}{dx} = \frac{1}{2}\frac{d}{dx}(px) - \frac{1}{2}\frac{d}{dx}(2x^2)$$

$$\frac{dA}{dx} = \frac{1}{2}(p) - \frac{2}{2}(2x)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dA}{dx} = \frac{p}{2} - (2x)$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dx} = \frac{p}{2} - (2x) = 0$$

$$2x = \frac{p}{2}$$

$x = \frac{p}{4}$

Now to check if this critical point will determine the largest rectangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{p}{2} - (2x)\right]$$

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{p}{2}\right) - \frac{d}{dx}(2x)$$

$$\frac{d^2A}{dx^2} = 0 - 2 = -2$$

[Since $\frac{d}{dx}$ (constant) = 0 and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = -2$ ....(4)

Now, consider the value of $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{p}{4}}$

$$\frac{d^2A}{dx^2} = -2 < 0$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{p}{4}} = -2 < 0$, so the function P is maximum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$$

$$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

$$\therefore\ y = \frac{p}{4}$$

As $= y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a largest rectangle which has a given perimeter is a square.

Question: 9

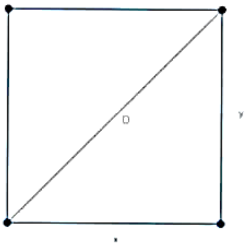
Solution:

Given,

- Rectangle with given perimeter.

Let us consider,

- ‘p’ as the fixed perimeter of the rectangle.
- ‘x’ and ‘y’ be the sides of the given rectangle.
- Diagonal of the rectangle, $D = \sqrt{x^2 + y^2}$. (using the hypotenuse formula)



Now as consider the perimeter of the rectangle,

p = 2(x +y)

p = 2x + 2y

$y = \frac{p-2x}{2}$ ----- (1)

Consider the diagonal of the rectangle,

$D = \sqrt{x^2 + y^2}$

Substituting (1) in the diagonal of the rectangle,

$$D = \sqrt{x^2 + \left(\frac{p-2x}{2}\right)^2}$$

[squaring both sides]

$Z = D^2 = x^2 + \left(\frac{p-2x}{2}\right)^2$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2 + \left(\frac{p-2x}{2}\right)^2\right]$$

$$\frac{dZ}{dx} = \frac{d}{dx}(x^2) + \frac{1}{4}\frac{d}{dx}[(p-2x)^2]$$

$$\frac{dZ}{dx} = 2x + \frac{1}{4}[2(p-2x)(-2)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

= 2x - p + 2x

$\frac{dZ}{dx} = 4x - p$ ....(3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 4x - p = 0$$

4x –p = 0

4x = p

$$x = \frac{p}{4}$$

<u>Now to check if this critical point will determine the minimum diagonal, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[4x - p]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(4x) - \frac{d}{dx}(p)$$

$= 4 + 0$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2Z}{dx^2} = 4$ ....(4)

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}}$

$$\frac{d^2Z}{dx^2} = 4 > 0$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{p}{4}} = 4 > 0$, so the function Z is minimum at $x = \frac{p}{4}$.

Now substituting $x = \frac{p}{4}$ in equation (1):

$$y = \frac{p - 2\left(\frac{p}{4}\right)}{2}$$

$$y = \frac{p - \frac{p}{2}}{2} = \frac{p}{4}$$

$$\therefore y = \frac{p}{4}$$

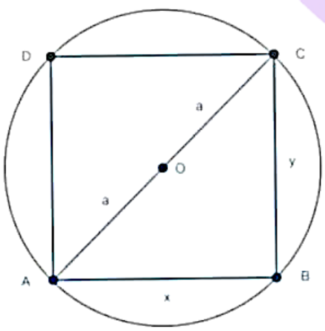
As $x = y = \frac{p}{4}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with minimum diagonal which has a given perimeter is a square.

Question: 10

Solution:

Given,

- Rectangle is of maximum perimeter.
- The rectangle is inscribed inside a circle.
- The radius of the circle is 'a'.



Let us consider,

- 'x' and 'y' be the length and breadth of the given rectangle.
- Diagonal $AC^2 = AB^2 + BC^2$ is given by $4a^2 = x^2 + y^2$ (as AC = 2a)
- Perimeter of the rectangle, P = 2(x+y)

Consider the diagonal,

$4a^2 = x^2 + y^2$

$y^2 = 4a^2 - x^2$

$y = \sqrt{4a^2 - x^2}$ ---- (1)

Now, perimeter of the rectangle, P

$P = 2x + 2y$

Substituting (1) in the perimeter of the rectangle.

$P = 2x + 2\sqrt{4a^2 - x^2}$ ....(2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[2x + 2\sqrt{4a^2 - x^2}\right]$$

$$\frac{dP}{dx} = \frac{d}{dx}(2x) + 2\frac{d}{dx}[\sqrt{4a^2 - x^2}]$$

$$\frac{dP}{dx} = 2 + 2[\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}} \text{.........(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4a^2 - x^2}} = 0$$

$$2 = \frac{2x}{\sqrt{4a^2 - x^2}}$$

$$\sqrt{4a^2 - x^2} = x$$

[squaring on both sides]

$4a^2 - x^2 = x^2$

$2x^2 = 4a^2$

$x^2 = 2a^2$

$x = \pm a\sqrt{2}$

$x = a\sqrt{2}$

[as x cannot be negative]

Now to check if this critical point will determine the maximum diagonal, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2 - \frac{2x}{\sqrt{4a^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(2) - \frac{d}{dx}\left(\frac{2x}{\sqrt{4a^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{4a^2 - x^2}\,\frac{d}{dx}(2x) - (2x)\frac{d}{dx}(\sqrt{4a^2 - x^2})}{(\sqrt{4a^2 - x^2})^2}\right]$$

[Since $\frac{d}{dx}$ (constant) $= 0$ and $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) - (2x)\frac{1}{2}(4a^2 - x^2)^{-\frac{1}{2}}(-2x)}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{4a^2 - x^2}\,(2) + (2x^2)(4a^2 - x^2)^{-\frac{1}{2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2\sqrt{4a^2 - x^2} + \frac{2x^2}{\sqrt{4a^2 - x^2}}}{4a^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{2(4a^2 - x^2) + 2x^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{8a^2}{(4a^2 - x^2)^{\frac{3}{2}}}\right] \text{ ----- (4)}$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - (a\sqrt{2})^2)^{\frac{3}{2}}}\right]$$

$$\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\left[\frac{8a^2}{(4a^2 - 2a^2)^{\frac{3}{2}}}\right] = -\frac{8a^2}{(2a^2)^{\frac{3}{2}}} = -\frac{8a^2}{2\sqrt{2}\,a^3} = -\frac{2\sqrt{2}}{a}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=a\sqrt{2}} = -\frac{2\sqrt{2}}{a} < 0$, so the function P is maximum at $x = a\sqrt{2}$.

Now substituting $x = a\sqrt{2}$ in equation (1):

$$y = \sqrt{4a^2 - (a\sqrt{2})^2}$$

$$y = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2}$$

$$\therefore y = a\sqrt{2}$$

As $x = y = a\sqrt{2}$ the sides of the taken rectangle are equal, we can clearly say that a rectangle with maximum perimeter which is inscribed inside a circle of radius ‘a’ is a square.

Question: 11

Solution:

Given,

• Sum of perimeter of square and circle.

Let us consider,

• ‘x’ be the side of the square.

- 'r' be the radius of the circle.
- Let 'p' be the sum of perimeters of square and circle.

$p = 4x + 2\pi r$

Consider the sum of the perimeters of square and circle.

$p = 4x + 2\pi r$

$4x = p - 2\pi r$

$x = \frac{p-2\pi r}{4}$ ---- (1)

Sum of the area of the circle and square is

$A = x^2 + \pi r^2$

Substituting (1) in the sum of the areas,

$$A = \left(\frac{p - 2\pi r}{4}\right)^2 + \pi r^2$$

$$A = \frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi p r] + \pi r^2 \text{ ....(2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dA}{dr} = \frac{d}{dr}\left[\frac{1}{16}[p^2 + 4\pi^2 r^2 - 4\pi p r] + \pi r^2\right]$$

$$\frac{dA}{dr} = \frac{1}{16}\frac{d}{dr}(p^2 + 4\pi^2 r^2 - 4\pi p r) + \frac{d}{dr}[\pi r^2]$$

$$\frac{dA}{dr} = \frac{1}{16}(0 + 8\pi^2 r - 4\pi p) + 2\pi r$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(\text{constant}) = 0$]

$$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r \text{ ....(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dA}{dr} = \frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r = 0$$

$$\left(\frac{\pi^2}{2} + 2\pi\right) r - \frac{\pi p}{4} = 0$$

$$r = \frac{\frac{\pi p}{4}}{\frac{\pi^2}{2} + 2\pi} = \frac{2\pi p}{4(\pi^2 + 4\pi)} = \frac{\pi p}{2(\pi^2 + 4\pi)}$$

$$r = \frac{\pi p}{2\pi(\pi + 4)} = \frac{p}{2(\pi + 4)}$$

$$r = \frac{p}{2(\pi + 4)}$$

<u>Now to check if this critical point will determine the least of the sum of the areas of square and circle, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2A}{dr^2} = \frac{d}{dx}\left[\frac{\pi^2 r}{2} - \frac{\pi p}{4} + 2\pi r\right]$$

$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{\pi^2 r}{2}\right) - \frac{d}{dr}\left(\frac{\pi p}{4}\right) + \frac{d}{dr}(2\pi r)$$

$$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} - 0 + 2\pi$$

[Since $\frac{d}{dx}$ (constant) = 0 and $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{d^2A}{dr^2} = \frac{\pi^2}{2} + 2\pi \text{ ----- (4)}$$

Now, consider the value of $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}}$

$$\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi$$

As $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{p}{2(\pi+4)}} = \frac{\pi^2}{2} + 2\pi > 0$, so the function A is minimum at $r = \frac{p}{2(\pi+4)}$.

Now substituting $r = \frac{p}{2(\pi+4)}$ in equation (1):

$$x = \frac{p - 2\pi\left(\frac{p}{2(\pi+4)}\right)}{4}$$

$$x = \frac{p(\pi+4) - \pi p}{4 \times (\pi+4)} = \frac{\pi p + 4p - \pi p}{4\pi + 16} = \frac{4p}{4(\pi+4)}$$

$$x = \frac{p}{\pi+4}$$

As the side of the square,

$$x = \frac{p}{\pi+4}$$

$$x = 2\left[\frac{p}{2(\pi+4)}\right] = 2r$$

[as $r = \frac{p}{2(\pi+4)}$]

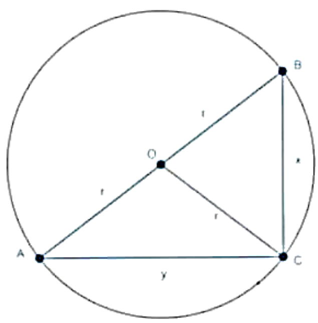
Therefore, side of the square, x = 2r = diameter of the circle.

Question: 12

Solution:

Given,

- A right angle triangle is inscribed inside the circle.
- The radius of the circle is given.



Let us consider,

- 'r' is the radius of the circle.
- 'x' and 'y' be the base and height of the right angle triangle.
- The hypotenuse of the $\Delta ABC = AB^2 = AC^2 + BC^2$

$AB = 2r$, $AC = y$ and $BC = x$

Hence,

$4r^2 = x^2 + y^2$

$y^2 = 4r^2 - x^2$

$y = \sqrt{4r^2 - x^2}$ --- (1)

Now, Area of the ΔABC is

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$A = \frac{1}{2} \times x \times y$$

Now substituting (1) in the area of the triangle,

$$A = \frac{1}{2} x (\sqrt{4r^2 - x^2})$$

[Squaring both sides]

$$Z = A^2 = \frac{1}{4} x^2 (4r^2 - x^2) \text{ ....(2)}$$

For finding the maximum / minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[\frac{1}{4} x^2 (4r^2 - x^2)\right]$$

$$\frac{dZ}{dx} = \frac{1}{4}\left[(4r^2 - x^2)\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(4r^2 - x^2)\right]$$

$$\frac{dZ}{dx} = \frac{1}{4}[(4r^2 - x^2) \times (2x) + x^2 (0 - 2x)]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and if u and v are two functions of x, then $\frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$]

$$\frac{dZ}{dx} = \frac{1}{4}[8r^2x - 2x^3 - 2x^3]$$

$$\frac{dZ}{dx} = \frac{1}{4}[8r^2x - 4x^3] = \frac{4x}{4}[2r^2 - x^2]$$

$$\frac{dZ}{dx} = 2r^2x - x^3 \text{ ....(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 2r^2x - x^3 = 0$$

$2r^2x = x^3$

$x^2 = 2r^2$

$x = \pm\sqrt{2r^2}$

$x = r\sqrt{2}$

[as the base of the triangle cannot be negative.]

Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2r^2x - x^3]$$

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}(2r^2x) - \frac{d}{dx}(x^3)$$

$$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2 \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now, consider the value of $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}}$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2 = 2r^2 - 6r^2 = -4r^2$$

As $\left(\frac{d^2Z}{dx^2}\right)_{x=r\sqrt{2}} = -4r^2 < 0$, so the function A is maximum at $x = r\sqrt{2}$.

Now substituting $x = r\sqrt{2}$ in equation (1):

$$y = \sqrt{4r^2 - (r\sqrt{2})^2}$$

$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = r\sqrt{2}$$

As $x = y = r\sqrt{2}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

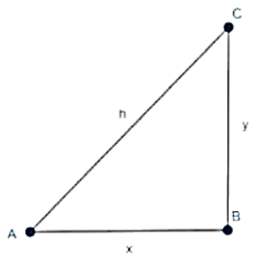
Hence the given triangle, which is inscribed in a circle, is an isosceles triangle with sides AC and BC equal.

Question: 13

Solution:

Given,

- A right angle triangle.
- Hypotenuse of the given triangle is given.



Let us consider,

- 'h' is the hypotenuse of the given triangle.
- 'x' and 'y' be the base and height of the right angle triangle.
- The hypotenuse of the $\Delta ABC = AC^2 = AB^2 + BC^2$

$AC = h$, $AB = x$ and $BC = y$

Hence,

$h^2 = x^2 + y^2$

$y^2 = h^2 - x^2$

$y = \sqrt{h^2 - x^2}$ --- (1)

Now, perimeter of the ΔABC is

$P = h + x + y$

Now substituting (1) in the area of the triangle,

$P = h + x + \sqrt{h^2 - x^2}$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dP}{dx} = \frac{d}{dx}\left[h + x + \sqrt{h^2 - x^2}\right]$$

$$\frac{dP}{dx} = \left[\frac{d}{dx}(h) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\sqrt{h^2 - x^2}\right)\right]$$

$$\frac{dP}{dx} = 0 + 1 + \frac{1}{2}\left(\frac{-2x}{\sqrt{h^2 - x^2}}\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}} \text{.........(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dP}{dx} = 1 - \frac{x}{\sqrt{h^2 - x^2}} = 0$$

$$\frac{x}{\sqrt{h^2 - x^2}} = 1$$

$$x = \sqrt{h^2 - x^2}$$

[squaring on both sides]

$$x^2 = h^2 - x^2$$

$$x^2 = \frac{h^2}{2}$$

$$x = \pm\sqrt{\frac{h^2}{2}}$$

$$x = \frac{h}{\sqrt{2}}$$

[as the base of the triangle cannot be negative.]

Now to check if this critical point will determine the maximum perimeter of the triangle, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[1 - \frac{x}{\sqrt{h^2 - x^2}}\right]$$

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$$\frac{d^2P}{dx^2} = \frac{d}{dx}(1) - \frac{d}{dx}\left(\frac{x}{\sqrt{h^2 - x^2}}\right)$$

$$\frac{d^2P}{dx^2} = 0 - \left[\frac{\sqrt{h^2 - x^2}\,\frac{d}{dx}(x) - x\,\frac{d}{dx}\left(\sqrt{h^2 - x^2}\right)}{\left(\sqrt{h^2 - x^2}\right)^2}\right]$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ if u and v are two functions of x, then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$]

$$\frac{d^2P}{dx^2} = -\left[\frac{\sqrt{h^2 - x^2}\,(1) - x\left(\frac{-2x}{2\sqrt{h^2 - x^2}}\right)}{h^2 - x^2}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{(\sqrt{h^2 - x^2})^2 + x^2}{h^2 - x^2\,\sqrt{h^2 - x^2}}\right] = -\left[\frac{h^2}{(h^2 - x^2)\,\sqrt{h^2 - x^2}}\right]$$

$$\frac{d^2P}{dx^2} = -\left[\frac{h^2}{(h^2 - x^2)^{\frac{3}{2}}}\right]$$

Now, consider the value of $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}}$

$$\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\left[\frac{h^2}{(h^2 - \left(\frac{h}{\sqrt{2}}\right)^2)^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{(\frac{h^2}{2})^{\frac{3}{2}}}\right] = -\left[\frac{h^2}{(\frac{h^2}{2})^{\frac{3}{2}}}\right] = -\frac{2^{\frac{3}{2}}}{h}$$

As $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -\frac{2^{\frac{3}{2}}}{h} < 0$, so the function A is maximum at $x = \frac{h}{\sqrt{2}}$.

Now substituting $x = \frac{h}{\sqrt{2}}$ in equation (1):

$$y = \sqrt{h^2 - \left(\frac{h}{\sqrt{2}}\right)^2}$$

$$y = \sqrt{\frac{h^2}{2}} = \frac{h}{\sqrt{2}}$$

As $x = y = \frac{h}{\sqrt{2}}$, the base and height of the triangle are equal, which means that two sides of a right angled triangle are equal,

Hence the given triangle is an isosceles triangle with sides AB and BC equal.

Question: 14

Solution:

Given,

- Perimeter of a triangle is 8 cm.
- One of the sides of the triangle is 3 cm.
- The area of the triangle is maximum.

Let us consider,

- 'x' and 'y' be the other two sides of the triangle.

Now, perimeter of the ΔABC is

8 = 3 + x + y

y = 8-3-x = 5-x

y = 5-x --- (1)

Consider the Heron's area of the triangle,

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s = \frac{a+b+c}{2}$

As perimeter = a + b+ c = 8

$$s = \frac{8}{2} = 4$$

Now Area of the triangle is given by

$$A = \sqrt{8\,(8-3)(8-x)(8-y)}$$

Now substituting (1) in the area of the triangle,

$$A = \sqrt{4(4-3)(4-x)(4-(5-x))}$$

$$A = \sqrt{4(4-x)(x-1)}$$

$$A = \sqrt{4(\,4x-4-x^2+x)} = \sqrt{4(\,5x-x^2-4)}$$

$$A = \sqrt{4(\,5x-x^2-4)}$$

[squaring on both sides]

$Z = A^2 = 4(5x - x^2 - 4)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.</u>

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\,[4(5x - x^2 - 4)]$$

$$\frac{dZ}{dx} = 4\,\frac{d}{dx}(5x) - 4\frac{d}{dx}\,(x^2) - 4\,\frac{d}{dx}\,(4)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$]

$$\frac{dZ}{dx} = 4\,(5) - 4(2x) - 0$$

$\frac{dZ}{dx} = 20 - 8x$........(3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 20 - 8x = 0$$

20 – 8x = 0

8x = 20

$$x = \frac{5}{2}$$

<u>Now to check if this critical point will determine the maximum area of the triangle, we need to check with second differential which needs to be negative</u>.

Consider differentiating the equation (3) with x:

$\frac{d^2Z}{dx^2} = \frac{d}{dx}[20 - 8x]$

$\frac{d^2Z}{dx^2} = -8$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

As $\left(\frac{d^2Z}{dx^2}\right)_{x=\frac{5}{2}} = -8 < 0$, so the function A is maximum at $x = \frac{5}{2}$.

Now substituting $x = \frac{5}{2}$ in equation (1):

y = 5 – 2.5

y = 2.5

As x = y = 2.5, two sides of the triangle are equal,

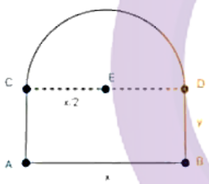
Hence the given triangle is an isosceles triangle with two sides equal to 2.5 cm and the third side equal to 3cm.

Question: 15

Solution:

Given,

- Window is in the form of a rectangle which has a semicircle mounted on it.
- Total Perimeter of the window is 10 metres.
- The total area of the window is maximum.



Let us consider,

- The breadth and height of the rectangle be 'x' and 'y'.
- The radius of the semicircle will be half of the base of the rectangle.

Given Perimeter of the window is 10 meters:

$$10 = (x + 2y) + \frac{1}{2}\left[2\pi\left(\frac{x}{2}\right)\right]$$

[as the perimeter of the window will be equal to one side (x) less to the perimeter of rectangle and the perimeter of the semicircle.]

$$10 = (x + 2y) + \left(\frac{\pi x}{2}\right)$$

From here,

$$2y = 10 - x - \left(\frac{\pi x}{2}\right) = \frac{20 - 2x - \pi x}{2}$$

$y = \frac{20-2x-\pi x}{4}$ ----- (1)

Now consider the area of the window,

Area of the window = area of the semicircle + area of the rectangle

$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + xy$

Substituting (1) in the area equation:

$A = \frac{1}{2}\left[\pi\left(\frac{x}{2}\right)^2\right] + x\left(\frac{20 - 2x - \pi x}{4}\right)$

$A = \frac{1}{8}[\pi x^2] + \left(\frac{20x - 2x^2 - \pi x^2}{4}\right)$

$A = \frac{\pi x^2 - 2\pi x^2 + 40x - 4x^2}{8}$

$A = \frac{1}{8}[x^2(\pi - 2\pi - 4) + 40x]$ ....(2)

For finding the maximum / minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to x:

$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{8}[x^2(\pi - 2\pi - 4) + 40x]\right]$

$\frac{dA}{dx} = \frac{1}{8}\frac{d}{dx}\left(x^2(\pi - 2\pi - 4)\right) + \frac{1}{8}\frac{d}{dx}(40x)$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dA}{dx} = \frac{1}{8}[2x(-\pi - 4)] + \frac{1}{8}(40)$

$\frac{dA}{dx} = \frac{1}{4}[x(-\pi - 4)] + 5$ ....(3)

To find the critical point, we need to equate equation (3) to zero.

$\frac{dA}{dx} = \frac{1}{4}[x(-\pi - 4)] + 5 = 0$

$\frac{1}{4}[x(-\pi - 4)] + 5 = 0$

$\frac{1}{4}[x(4 + \pi)] = 5$

$x(4 + \pi) = 20$

$x = \frac{20}{(4 + \pi)}$

Now to check if this critical point will determine the maximum area of the window, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}[x(-\pi - 4)] + 5\right]$

$\frac{d^2A}{dx^2} = \frac{d}{dx}[x(-\pi - 4)] + \frac{d}{dx}(5)$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{d^2A}{dx^2} = (-\pi - 4)(1) + 0 = -(\pi + 4)$ ....(4)

As $\left(\frac{d^2A}{dx^2}\right)_{x = \frac{20}{(4+\pi)}} = -(\pi + 4) < 0$, so the function A is maximum at $x = \frac{20}{(4+\pi)}$.

Now substituting $x = \frac{20}{(4+\pi)}$ in equation (1):

$$y = \frac{20 - \left(\frac{20}{(4+\pi)}\right)(\pi + 2)}{4}$$

$$y = \frac{20(4+\pi) - (20)(\pi + 2)}{4\,(4+\pi)} = \frac{20\,[4 + \pi - \pi - 2]}{4(4+\pi)} = \frac{20 \times 2}{4(4+\pi)}$$

$$y = \frac{5 \times 2}{(4+\pi)} = \frac{10}{(4+\pi)}$$

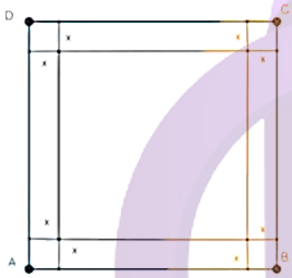
Hence the given window with maximum area has breadth, $x = \frac{20}{(4+\pi)}$ and height, $y = \frac{10}{(4+\pi)}$.

Question: 16

Solution:

Given,

- Side of the square piece is 12 cms.
- the volume of the formed box is maximum.



Let us consider,

- 'x' be the length and breadth of the piece cut from each vertex of the piece.
- Side of the box now will be (12-2x)
- The height of the new formed box will also be 'x'.

Let the volume of the newly formed box is :

$V = (12-2x)^2 \times (x)$

$V = (144 + 4x^2 - 48x)\ x$

$V = 4x^3 - 48x^2 + 144x$ ------(1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function f(x) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx} = \frac{d}{dx}\left[4x^3 - 48x^2 + 144x\right]$$

$$\frac{dV}{dx} = 12x^2 - 96x + 144 \text{ ....(2)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx} = 12x^2 - 96x + 144 = 0$$

$x^2 - 8x + 12 = 0$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(12)}}{2(1)} = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm \sqrt{16}}{2}$$

$$x == \frac{8 \pm 4}{2}$$

x = 6 or x =2

x= 2

[as x = 6 is not a possibility, because 12-2x = 12-12= 0]

Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2} = \frac{d}{dx}[\,12x^2 - 96x + 144]$$

$$\frac{d^2V}{dx^2} = 24x - 96 \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=2} = 24\,(2) - 96 = 48 - 96 = -48$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=2} = -48 < 0$, so the function A is maximum at x = 2

Now substituting x = 2 in 12 – 2x, the side of the considered box:

Side = 12-2x = 12 - 2(2) = 12-4 = 8cms

Therefore side of wanted box is 8cms and height of the box is 2cms.

Now, the volume of the box is

$V = (8)^2 \times 2 = 64 \times 2 = 128cm^3$

Hence maximum volume of the box formed by cutting the given 12cms sheet is $128cm^3$ with 8cms side and 2cms height.

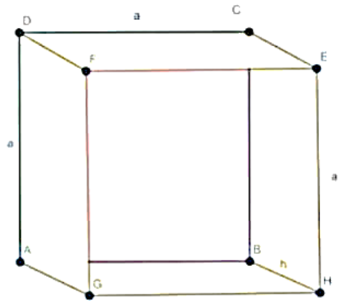
Question: 17

An open box with

Solution:

Given,

- The open box has a square base
- The area of the box is c^2 square units.
- The volume of the box is maximum.



Let us consider,

- The side of the square base of the box be ‘a’ units. (pink coloured in the figure)
- The breadth of the 4 sides of the box will also be ‘a’units (skin coloured part).
- The depth of the box or the length of the sides be ‘h’ units (skin coloured part).

Now, the area of the box =

(area of the base) + 4 (area of each side of the box)

So as area of the box is given c^2,

$c^2 = a^2 + 4ah$

$h = \frac{c^2 - a^2}{4a}$ ---- (1)

Let the volume of the newly formed box is :

$V = (a)^2 \times (h)$

[substituting (1) in the volume formula]

$$V = a^2 \times \left(\frac{c^2 - a^2}{4a}\right)$$

$$V = \left(\frac{ac^2 - a^3}{4}\right) \text{.........(2)}$$

For finding the maximum / minimum of given function, we can find it by differentiating it with a and then equating it to zero. This is because if the function f(a) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with respect to a:

$$\frac{dV}{da} = \frac{d}{da}\left[\left(\frac{ac^2 - a^3}{4}\right)\right]$$

$$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4} \text{.........(3)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{da} = \frac{c^2}{4} - \frac{3a^2}{4} = 0$$

$c^2 - 3a^2 = 0$

$$a^2 = \frac{c^2}{3}$$

$$a = \pm\sqrt{\frac{c^2}{3}}$$

$a = \frac{c}{\sqrt{3}}$

[as ‘a’ cannot be negative]

Now to check if this critical point will determine the maximum Volume of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{da^2} = \frac{d}{dx}\left[\frac{c^2}{4} - \frac{3a^2}{4}\right]$$

$$\frac{d^2V}{da^2} = 0 - \frac{3 \times 2 \times a}{4} = -\frac{3a}{2} \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -\frac{3\left(\frac{c}{\sqrt{3}}\right)}{2} = -\frac{c\sqrt{3}}{2}$$

As $\left(\frac{d^2V}{da^2}\right)_{a=\frac{c}{\sqrt{3}}} = -48 - \frac{c\sqrt{3}}{2} < 0$, so the function V is maximum at $a = \frac{c}{\sqrt{3}}$

Now substituting a in equation (1)

$$h = \frac{c^2 - \left(\frac{c}{\sqrt{3}}\right)^2}{4\left(\frac{c}{\sqrt{3}}\right)} = \frac{\frac{2c^2}{3}}{\frac{4c}{\sqrt{3}}} = \frac{c\sqrt{3}}{6} = \frac{c}{2\sqrt{3}}$$

$$\therefore h = \frac{c}{2\sqrt{3}}$$

Therefore side of wanted box has a base side, $a = \frac{c}{\sqrt{3}}$ is and height of the box, $h = \frac{c}{2\sqrt{3}}$.

Now, the volume of the box is

$$V = \left(\frac{c}{\sqrt{3}}\right)^2 \times \left(\frac{c}{2\sqrt{3}}\right)$$

$$V = \frac{c^2}{3} \times \left(\frac{c}{2\sqrt{3}}\right) = \frac{c^3}{6\sqrt{3}}$$

$$\therefore V = \frac{c^3}{6\sqrt{3}}$$

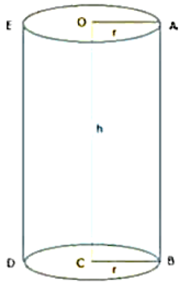
Question: 18

A cylindrical can

Solution:

Given,

- The can is cylindrical with a circular base
- The volume of the cylinder is 1 litre = 1000 cm^2.
- The surface area of the box is minimum as we need to find the minimum dimensions.



Let us consider,

- The radius base and top of the cylinder be ‘r’ units. (skin coloured in the figure)
- The height of the cylinder be ‘h’units.
- As the Volume of cylinder is given, $V = 1000 cm^3$

The Volume of the cylinder= $\pi r^2 h$

$1000 = \pi r^2 h$

$h = \frac{1000}{\pi r^2}$ ---- (1)

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi rh$

$S = 2\pi r^2 + 2\pi rh$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$S = 2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right] \text{.........(2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function f(r) has a maximum/minimum at a point c then f’(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\left[\pi r^2 + \left(\frac{1000}{r}\right)\right]\right]$$

$$\frac{dS}{dr} = 2\,(2\pi r) + \left(\frac{1000}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) \text{.........(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2\,(2\pi r) - 2\left(\frac{1000}{r^2}\right) = 0$$

$$2\pi r = \frac{1000}{r^2}$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2(2\pi r) - 2\left(\frac{1000}{r^2}\right)\right]$$

$$\frac{d^2S}{dr^2} = 4\pi - \frac{2\times1000\times(-2)}{r^3} = 4\pi + \frac{4000}{r^3} \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 4\pi + \frac{4000}{\left(\sqrt[3]{\frac{500}{\pi}}\right)^3} = 4\pi + \frac{4000\times\pi}{500} = 4\pi + 8\pi = 12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{500}{\pi}}} = 12\pi > 0$, so the function S is minimum at $r = \sqrt[3]{\frac{500}{\pi}}$

Now substituting r in equation (1)

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = \frac{1000}{\pi^{\frac{1}{3}}(500)^{\frac{2}{3}}}$$

$$\therefore h = \frac{1000}{\pi^{\frac{1}{3}}(500)^{\frac{2}{3}}}$$

Therefore the radius of base of the cylinder, $r = \sqrt[3]{\frac{500}{\pi}}$ and height of the cylinder, $h = \frac{1000}{\pi^{\frac{1}{3}}(500)^{\frac{2}{3}}}$ where the surface area of the cylinder is minimum.

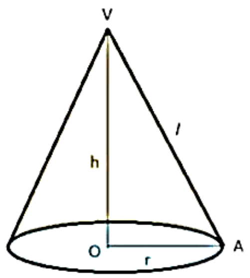
Question: 19

Show that the rig

Solution:

Given,

- The volume of the cone.
- The cone is right circular cone.
- The cone has least curved surface.



Let us consider,

- The radius of the circular base be ‘r’ cms.
- The height of the cone be ‘h’ cms.

- The slope of the cone be 'l' cms.

Given the Volume of the cone = $\pi r^2 l$

$$V = \frac{\pi r^2 h}{3}$$

$$h = \frac{3v}{\pi r^2} \text{ ----- (1)}$$

The Surface area cylinder is = πrl

$$S = \pi rl$$

$$S = \pi r\,(\sqrt{h^2 + r^2})$$

[substituting (1) in the Surface area formula]

$$S = \pi r \left[\sqrt{\left(\frac{3V}{\pi r^2}\right)^2 + r^2}\right]$$

[squaring on both sides]

$$Z = S^2 = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

$$Z = \pi^2 \left(\frac{9V^2}{\pi^2 r^2} + r^4\right) \text{----- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function Z has a maximum/minimum at a point c then Z'(c) = 0</u>.

Differentiating the equation (2) with respect to r:

$$\frac{dZ}{dr} = \frac{d}{dr}\left[\pi^2\left(\frac{9V^2}{\pi^2 r^2} + r^4\right)\right]$$

$$\frac{dZ}{dr} = \pi^2\left(\frac{9V^2}{\pi^2}\right)\frac{d}{dr}\left(\frac{1}{r^2}\right) + \pi^2\frac{d}{dr}(r^4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3}\right) + \pi^2 (4r^3) \text{.........(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dr} = \left(\frac{-18V^2}{r^3}\right) + \pi^2 (4r^3) = 0$$

$$\pi^2 (4r^3) = \frac{18V^2}{r^3}$$

$$2\pi^2 r^6 = 9V^2 \text{ ---- (4)}$$

<u>Now to check if this critical point will determine the minimum surface area of the cone, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with r:

$$\frac{d^2Z}{dr^2} = \frac{d}{dr}\left[\left(\frac{-18V^2}{r^3}\right) + \pi^2 (4r^3)\right]$$

$$\frac{d^2Z}{dr^2} = \frac{-18V^2(-3)}{r^4} + \pi^2 (4 \times 3r^2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{d^2Z}{dr^2} = \frac{54V^2}{r^4} + \pi^2 (12 r^2)$$

Now let us find the value of

$$\left(\frac{d^2Z}{dr^2}\right) = \frac{54V^2}{r^4} + \pi^2 (12 r^2) > 0$$

As $\left(\frac{d^2Z}{dr^2}\right) > 0$, so the function $Z = S^2$ is minimum

Now consider, the equation (4),

$$9V^2 = 2\pi^2 r^6$$

Now substitute the volume of the cone formula in the above equation.

$$9\left(\frac{\pi r^2 h}{3}\right)^2 = 2\pi^2 r^6$$

$$\pi^2 r^4 h^2 = 2\,\pi^2 r^6$$

$$2r^2 = h^2$$

$$h = r\sqrt{2}$$

Hence, the relation between h and r of the cone is proved when S is the minimum.

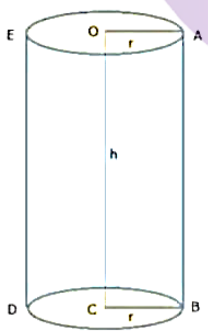
Question: 20

Find the radius o

Solution:

Given,

- The closed is cylindrical can with a circular base and top.
- The volume of the cylinder is 1 litre = 100 cm^3.
- The surface area of the box is minimum.



Let us consider,

- The radius base and top of the cylinder be 'r' units. (skin coloured in the figure)
- The height of the cylinder be 'h'units.
- As the Volume of cylinder is given, $V = 100cm^3$

The Volume of the cylinder= $\pi r^2 h$

$$100 = \pi r^2 h$$

$$h = \frac{100}{\pi r^2} \text{ ---- } (1)$$

The Surface area cylinder is = area of the circular base + area of the circular top + area of the cylinder

$S = \pi r^2 + \pi r^2 + 2\pi rh$

$S = 2\ \pi r^2 + 2\pi rh$

[substituting (1) in the volume formula]

$$S = 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right)$$

$$S = 2\left[\pi r^2 + \left(\frac{100}{r}\right)\right] \text{........(2)}$$

For finding the maximum / minimum of given function, we can find it by differentiating it with r and then equating it to zero. This is because if the function f(r) has a maximum/minimum at a point c then f'(c) = 0.

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\left[\pi r^2 + \left(\frac{100}{r}\right)\right]\right]$$

$$\frac{dS}{dr} = 2\,(2\pi r) + \left(\frac{100}{r^2}\right)(-1)$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{100}{r^2}\right) \text{.........(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\,(2\pi r) - 2\left(\frac{100}{r^2}\right) = 0$$

$$2\,(2\pi r) - 2\left(\frac{100}{r^2}\right) = 0$$

$$2\pi r = \frac{100}{r^2} \text{ ---- (4)}$$

Now to check if this critical point will determine the minimum surface area of the box, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\,(2\pi r) - 2\left(\frac{100}{r^2}\right)\right]$$

$$\frac{d^2S}{dr^2} = 4\pi - \frac{2\times 100\times(-2)}{r^3} = 4\pi + \frac{400}{r^3} \text{ ----- (5)}$$

[Since $\frac{d}{dx}\,(x^n) = nx^{n-1}$ and $\frac{d}{dx}\,(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}} = 4\pi + \frac{400}{\left(\sqrt[3]{\frac{50}{\pi}}\right)^3} = 4\pi + \frac{400\ \times\ \pi}{50} = 4\pi + 8\pi = 12\pi$$

As $\left(\frac{d^2S}{dr^2}\right)_{r=\sqrt[3]{\frac{50}{\pi}}} = 12\pi > 0$, so the function S is minimum at $r = \sqrt[3]{\frac{50}{\pi}}$

As S is minimum from equation (4)

$$2\pi r = \frac{100}{r^2}$$

$$2\pi r = \frac{V}{r^2}$$

$$V = 2\pi r^3$$

Now in equation (1) we have,

$$h = \frac{V}{\pi r^2} = \frac{2\pi r^3}{\pi r^2}$$

h = 2r = diameter

Therefore when the total surface area of a cone is minimum, then height of the cone is equal to twice the radius or equal to its diameter.

Question: 21

Show that the hei

Solution:

Let r be the radius of the base and h the height of a cylinder.

The surface area is given by,

$S = 2\pi r^2 + 2\pi rh$

$h = \frac{S - 2\pi r^2}{2\pi r}$.........(1)

Let V be the volume of the cylinder.

Therefore, $V = \pi r^2 h$

$V = \pi r^2 \left(\frac{S - 2\pi r^3}{2\pi r}\right)$.......Using equation 1

$$V = \frac{Sr - 2\pi r^3}{2}$$

Differentiating both sides w.r.t r, we get,

$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$..........(2)

For maximum or minimum, we have,

$$\frac{dV}{dr} = 0$$

$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$

$\Rightarrow S = 6\pi r^2$

$2\pi r^2 + 2\pi rh = 6\pi r^2$

$h = 2r$

Differentiating equation 2, with respect to r to check for maxima and minima, we get,

$$\frac{d^2V}{dr^2} = -6\pi r < 0$$

Hence, V is maximum when h = 2r or h = diameter

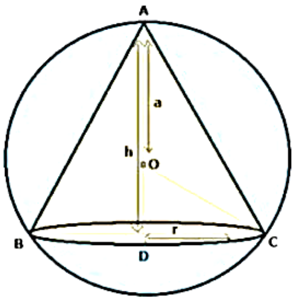
Question: 22

Prove that the vo

Solution:

Given,

- Volume of the sphere.
- Volume of the cone.
- Cone is inscribed in the sphere.
- Volume of cone is maximum.



Let us consider,

- The radius of the sphere be ‘a’ units.
- Volume of the inscribed cone be ‘V’.
- Height of the inscribed cone be ‘h’.
- Radius of the base of the cone is ‘r’.

Given volume of the inscribed cone is,

$$V = \frac{\pi r^2 h}{3}$$

Consider OD = (AD-OA) =(h-a)

Now let $OC^2 = OD^2 + DC^2$, here OC = a, OD = (h-a), DC = r,

So $a^2 = (h-a)^2 + r^2$

$r^2 = a^2 - (h^2 + a^2 - 2ah)$

$r^2 = h(2a - h)$------(1)

Let us consider the volume of the cone:

$$V = \frac{1}{3}(\pi r^2 h)$$

Now substituting (1) in the volume formula,

$$V = \frac{1}{3}(\pi h(2a - h)h)$$

$$V = \frac{1}{3}(2\pi h^2 a - \pi h^3) \text{ ---- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. This is because if the function V(r) has a maximum/minimum at a point c then V’(c) = 0.

Differentiating the equation (2) with respect to h:

$$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{1}{3}(2\pi h^2 a - \pi h^3)\right]$$

$$\frac{dV}{dh} = \frac{1}{3}(2\pi a)(2h) - \frac{1}{3}(\pi)(3h^2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dV}{dh} = \frac{1}{3}[4\pi ah - 3\pi h^2] \text{ ....(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{dh} = \frac{1}{3}[4\pi ah - 3\pi h^2] = 0$$

$4\pi ah - 3\pi h^2 = 0$

h(4πa-3πh)= 0

$h = 0$ (or) $h = \frac{4\pi a}{3\pi} = \frac{4a}{3}$

$$h = \frac{4a}{3}$$

[as h cannot be zero]

Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with h:

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[\frac{1}{3}[\,4\pi ah - 3\pi h^2\,]\right]$$

$\frac{d^2V}{dh^2} = \frac{1}{3}[\,4\pi a - (3\pi)(2h)] = \frac{\pi}{3}[4a - 6h]$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{3}} = \frac{\pi}{3}\left[4a - 6\left(\frac{4a}{3}\right)\right] = \frac{4a\pi}{3}[1-2] = -\frac{4a\pi}{3}$$

As $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{4a}{3}} = -\frac{4a\pi}{3} < 0$, so the function V is maximum at $h = \frac{4a}{3}$

Substituting h in equation (1)

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \left(\frac{4a}{3}\right)\left(2a - \frac{4a}{3}\right)$$

$$r^2 = \frac{8a^2}{9}$$

As V is maximum, substituting h and r in the volume formula:

$$V = \frac{1}{3}\pi\left(\frac{8a^2}{9}\right)\left(\frac{4a}{3}\right)$$

$$V = \frac{8}{27}\left(\frac{4}{3}\pi a^3\right)$$

$$V = \frac{8}{27}(\text{volume of the sphere})$$

Therefore when the volume of a inscribed cone is maximum, then it is equal to $\frac{8}{27}$ times of the volume of the sphere in which it is inscribed.

Question: 23

Which fraction ex

Solution:

Given,

The pth power of a number exceeds by a fraction to be the greatest.

Let us consider,

- 'x' be the required fraction.
- The greatest number will be $y = x - x^p$---------(1)

For finding the maximum/ minimum of given function, we can find it by differentiatin and then equating it to zero. This is because if the function y(x)has a maximum/minin point c then y'(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}(x - x^p)$$

$\frac{dy}{dx} = 1 - px^{p-1}$ ---- (2)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dy}{dx} = 1 - px^{p-1} = 0$$

$1 = px^{p-1}$

$$x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$$

Now to check if this critical point will determine the if the number is the greatest, we need to check with second differential which needs to be negative.

Consider differentiating the equation (2) with x:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[1 - px^{p-1}]$$

$\frac{d^2y}{dx^2} = -p(p-1)x^{p-2}$ ----- (3)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2}$$

As $\left(\frac{d^2y}{dx^2}\right)_{x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}} = -p(p-1)\left(\left(\frac{1}{p}\right)^{\frac{1}{p-1}}\right)^{p-2} < 0$, so the number y is greatest at $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

Hence, the y is the greatest number and exceeds by a fraction $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

Question: 24

Find the point on

Solution:

Given,

- A point is present on a curve $y^2 = 4x$
- The point is near to the point (2,-8)

Let us consider,

- The co-ordinates of the point be P(x,y)
- As the point P is on the curve, then $y^2 = 4x$

$$x = \frac{y^2}{4}$$

- The distance between the points is given by,

$D^2 = (x-2)^2 + (y+8)^2$

$D^2 = x^2 - 4x + 4 + y^2 + 64 + 16y$

Substituting x in the distance equation

$$D^2 = \left(\frac{y^2}{4}\right)^2 - 4\left(\frac{y^2}{4}\right) + y^2 + 16y + 68$$

$$Z = D^2 = \frac{y^4}{16} + 16y + 68 \text{ ---- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with y and then equating it to zero. This is because if the function Z(x) has a maximum/minimum at a point c then Z'(c) = 0.

Differentiating the equation (2) with respect to y:

$$\frac{dZ}{dy} = \frac{d}{dy}\left(\frac{y^4}{16} + 16y + 68\right)$$

$$\frac{dZ}{dy} = \frac{4y^3}{16} + 16 = \frac{y^3}{4} + 16 \text{ ---- } (2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dZ}{dy} = \frac{y^3}{4} + 16 = 0$$

$y^3 + 64 = 0$

$(y + 4)(y^2 - 4y + 16) = 0$

$(y+4) = 0$ (or) $y^2 - 4y + 16 = 0$

$y = -4$

(as the roots of the $y^2 - 4y + 16$ are imaginary)

Now to check if this critical point will determine the distance is mimimum, we need to check with second differential which needs to be positive.

Consider differentiating the equation (2) with y:

$$\frac{d^2Z}{dy^2} = \frac{d}{dy}\left[\frac{y^3}{4} + 16\right]$$

$$\frac{d^2Z}{dy^2} = \frac{3y^2}{4} \text{ ----- } (3)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = \frac{3}{4}(-4)^2 = 12$$

As $\left(\frac{d^2Z}{dy^2}\right)_{y=-4} = 12 > 0$, so the Distance D^2 is minimum at $y = -4$

Now substituting y in x, we have

$$x = \frac{(-4)^2}{4} = 4$$

So, the point P on the curve $y^2 = 4x$ is (4,-4) which is at nearest from the (2,-8)

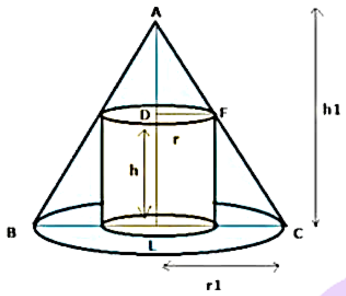
Question: 25

A right circular

Solution:

Given,

- A right circular cylinder is inscribed inside a cone.
- The curved surface area is maximum.



Let us consider,

- 'r_1' be the radius of the cone.
- 'h_1' be the height of the cone.
- 'r' be the radius of the inscribed cylinder.
- 'h' be the height of the inscribed cylinder.

DF = r, and AD = AL – DL = $h_1 - h$

Now, here ΔADF and ΔALC are similar,

Then

$$\frac{AD}{AL} = \frac{DF}{LC} \Rightarrow \frac{h_1 - h}{h_1} = \frac{r}{r_1}$$

$$h_1 - h = \frac{rh_1}{r_1}$$

$$h = h_1 - \frac{rh_1}{r_1} = h_1\left(1 - \frac{r}{r_1}\right)$$

$$h = h_1\left(1 - \frac{r}{r_1}\right) \text{ ----- (1)}$$

Now let us consider the curved surface area of the cylinder,

$S = 2\pi rh$

Substituting h in the formula,

$$S = 2\pi r\left[h_1\left(1 - \frac{r}{r_1}\right)\right]$$

$$S = 2\pi rh_1 - \frac{2\pi h_1 r^2}{r_1} \text{ ---- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with r and then equating it to zero. <u>This is because if the function S(r) has a maximum/minimum at a point c then S'(c) = 0.</u>

Differentiating the equation (2) with respect to r:

$$\frac{dS}{dr} = \frac{d}{dr}\left[2\pi rh_1 - \frac{2\pi h_1 r^2}{r_1}\right]$$

$$\frac{dS}{dr} = 2\pi h_1 - \frac{2\pi h_1 (2r)}{r_1}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1} \ldots\ldots\ldots(3)$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dr} = 2\pi h_1 - \frac{4\pi h_1 r}{r_1} = 0$$

$$\frac{4\pi h_1 r}{r_1} = 2\pi h_1$$

$$r = \frac{2\pi h_1 r_1}{4\pi h_1}$$

$$r = \frac{r_1}{2}$$

Now to check if this critical point will determine the maximum volume of the inscribed cylinder, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with r:

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[2\pi h_1 - \frac{4\pi h_1 r}{r_1}\right]$$

$$\frac{d^2S}{dr^2} = 0 - \frac{4\pi h_1}{r_1} = -\frac{4\pi h_1}{r_1} \text{ ----- } (4)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left.\frac{d^2S}{dr^2}\right|_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1}$$

As $\left.\frac{d^2S}{dr^2}\right|_{r=\frac{r_1}{2}} = -\frac{4\pi h_1}{r_1} < 0$, so the function S is maximum at $r = \frac{r_1}{2}$

Substituting r in equation (1)

$$h = h_1\left(1 - \frac{\frac{r_1}{2}}{r_1}\right)$$

$$h = h_1\left(1 - \frac{1}{2}\right) = \frac{h_1}{2} \text{ --- } (5)$$

As S is maximum, from (5) we can clearly say that $h_1 = 2h$ and

$r_1 = 2r$

this means the radius of the cone is twice the radius of the cylinder or equal to diameter of the cylinder.

Question: 26

Show that the sur

Solution:

Given,

- Closed cuboid has square base.
- The volume of the cuboid is given.

- Surface area is minimum.

Let us consider,

- The side of the square base be 'x'.
- The height of the cuboid be 'h'.
- The given volume, $V = x^2h$

$h = \frac{V}{x^2}$ ----- (1)

Consider the surface area of the cuboid,

Surface Area =

2(Area of the square base) + 4(areas of the rectangular sides)

$S = 2x^2 + 4xh$

Now substitute (1) in the Surface Area formula

$$S = 2x^2 + 4x\left(\frac{V}{x^2}\right)$$

$$S = 2x^2 + \left(\frac{4V}{x}\right) \text{ ----- (2)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function S(x) has a maximum/minimum at a point c then S'(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dS}{dx} = \frac{d}{dx}\left[2x^2 + \left(\frac{4V}{x}\right)\right]$$

$$\frac{dS}{dx} = 2\,(2x) + 4V\left(\frac{-1}{x^2}\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2} \text{ ....(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2} = 0$$

$$4x = \frac{4V}{x^2}$$

$$x^3 = V$$

Now to check if this critical point will determine the minimum surface area, we need to check with second differential which needs to be positive.

Consider differentiating the equation (3) with x:

$$\frac{d^2S}{dx^2} = \frac{d}{dx}\left[4x - \frac{4V}{x^2}\right]$$

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} \text{ ----- (4)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 4 + \frac{8V}{V} = 12$$

As $\frac{d^2S}{dx^2}_{x=V^{\frac{1}{3}}} = 12 > 0$, so the function S is minimum at $x = \sqrt[3]{V}$

Substituting x in equation (1)

$$h = \frac{V}{x^2} = \frac{x^3}{x^2} = x$$

h = x

As S is minimum and h = x, this means that the cuboid is a cube.

Question: 27

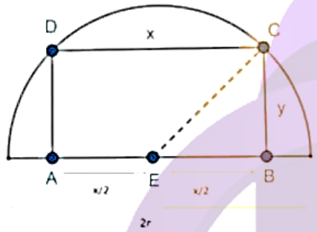
A rectangle is in

Solution:

Given,

• Radius of the semicircle is ‘r’.

• Area of the rectangle is maximum.



Let us consider,

• The base of the rectangle be ‘x’ and the height be ‘y’.

Consider the ΔCEB,

$CE^2 = EB^2 + BC^2$

As CE = r, EB = $\frac{x}{2}$ and CB = y

$$r^2 = \left(\frac{x}{2}\right)^2 + y^2$$

$$y^2 = r^2 - \left(\frac{x}{2}\right)^2 \text{ ---- } (1)$$

Now the area of the rectangle is

A = x × y

Squaring on both sides

$A^2 = x^2 y^2$

Substituting (1) in the above Area equation

$$A^2 = x^2 \left[r^2 - \left(\frac{x}{2}\right)^2\right]$$

$$Z = A^2 = x^2r^2 - x^2\frac{x^2}{4} = x^2r^2 - \frac{x^4}{4} \text{ ----- } (2)$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function Z(x) has a maximum/minimum at a point c then Z’(c) = 0.

Differentiating the equation (2) with respect to x:

$$\frac{dZ}{dx} = \frac{d}{dx}\left[x^2r^2 - \frac{x^4}{4}\right]$$

$$\frac{dZ}{dx} = r^2(2x) - \frac{4x^3}{4}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dZ}{dx} = 2xr^2 - x^3$ ....(3)

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dZ}{dx} = 2xr^2 - x^3 = 0$$

$x(2r^2 - x^2) = 0$

$x = 0$ (or) $x^2 = 2r^2$

$x = 0$ (or) $x = r\sqrt{2}$

$x = r\sqrt{2}$

[as x cannot be zero]

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2Z}{dx^2} = \frac{d}{dx}[2xr^2 - x^3]$$

$\frac{d^2Z}{dx^2} = 2r^2 - 3x^2$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left.\frac{d^2Z}{dx^2}\right|_{x=r\sqrt{2}} = 2r^2 - 3(r\sqrt{2})^2 = 2r^2 - 6r^2 = -4r^2$$

As $\left.\frac{d^2Z}{dx^2}\right|_{x=r\sqrt{2}} = -4r^2 < 0$, so the function Z is maximum at $x = r\sqrt{2}$

Substituting x in equation (1)

$$y^2 = r^2 - \left(\frac{r\sqrt{2}}{2}\right)^2 = r^2 - \frac{r^2}{2} = \frac{r^2}{2}$$

$$y = \sqrt{\frac{r^2}{2}} = \frac{r}{\sqrt{2}} = \frac{r\sqrt{2}}{2}$$

As the area of the rectangle is maximum, and $x = r\sqrt{2}$ and $y = \frac{r\sqrt{2}}{2}$

So area of the rectangle is

$$A = r\sqrt{2} \times \frac{r\sqrt{2}}{2}$$

$A = r^2$

Hence the maximum area of the rectangle inscribed inside a semicircle is r^2 square units.

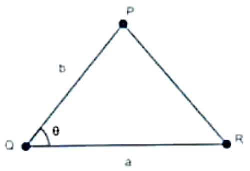
Question: 28

Two sides of a tr

Solution:

Given,

- The length two sides of a triangle are 'a' and 'b'
- Angle between the sides 'a' and 'b' is θ.
- The area of the triangle is maximum.



Let us consider,

The area of the ΔPQR is given be

$A = \frac{1}{2}$ ab sinθ ---- (1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with θ and then equating it to zero. This is because if the function A (θ) has a maximum/minimum at a point c then A'(c) = 0.

Differentiating the equation (1) with respect to θ:

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\left[\frac{1}{2}\text{ ab } \sin\theta\right]$$

$$\frac{dA}{d\theta} = \frac{1}{2}\text{ ab}\cos\theta \text{ ---- (2)}$$

[Since $\frac{d}{dx}(\sin\theta) = \cos\theta$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dA}{d\theta} = \frac{1}{2}\text{ ab}\cos\theta = 0$$

Cos θ = 0

$$\theta = \frac{\pi}{2}$$

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (2) with θ :

$$\frac{d^2A}{d\theta^2} = \frac{d}{d\theta}\left[\frac{1}{2}\text{ ab}\cos\theta\right]$$

$$\frac{d^2A}{d\theta^2} = -\frac{1}{2}\text{ ab}\sin\theta \text{ ----- (2)}$$

[Since $\frac{d}{dx}(\cos\theta) = -\sin\theta$]

Now let us find the value of

$$\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\text{ ab}\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2}\text{ ab}$$

As $\frac{d^2A}{d\theta^2}_{\theta=\frac{\pi}{2}} = -\frac{1}{2}\text{ ab} < 0$, so the function A is maximum at $\theta = \frac{\pi}{2}$

As the area of the triangle is maximum when $\theta = \frac{\pi}{2}$

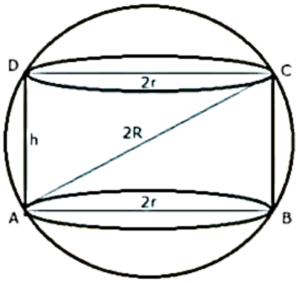
Question: 29

Show that the max

Solution:

Given,

- Radius of the sphere is $5\sqrt{3}$.
- Volume of cylinder is maximum.



Let us consider,

- The radius of the sphere be 'R' units.
- Volume of the inscribed cylinder be 'V'.
- Height of the inscribed cylinder be 'h'.
- Radius of the cylinder is 'r'.

Now let $AC^2 = AB^2 + BC^2$, here AC = 2R, AB =2r, BC = h,

So $4R^2 = 4r^2 + h^2$

$r^2 = \frac{1}{4}[4R^2 - h^2]$ ----- (1)

Let us consider, the volume of the cylinder:

$V = \pi r^2 h$

Now substituting (1) in the volume formula,

$V = \pi h\left(\frac{1}{4}[4R^2 - h^2]\right)$

$V = \frac{\pi}{4}(4R^2h - h^3)$ ---- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with h and then equating it to zero. <u>This is because if the function V(h) has a maximum/minimum at a point c then V'(c) = 0.</u>

Differentiating the equation (2) with respect to h:

$$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{\pi}{4}(4R^2h - h^3)\right]$$

$$\frac{dV}{dh} = \frac{4R^2\pi}{4} - \frac{\pi}{4}(3h^2)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4} \text{..........(3)}$$

To find the critical point, we need to equate equation (3) to zero.

$$\frac{dV}{dh} = R^2\pi - \frac{3h^2\pi}{4} = 0$$

$3h^2\pi = 4R^2\pi$

$$h^2 = \frac{4}{3}R^2 = \frac{4}{3}(5\sqrt{3})^2 = \frac{4}{3}(25 \times 3) = 100$$

h = 10

[as h cannot be negative]

Now to check if this critical point will determine the maximum volume of the inscribed cone, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with h:

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[R^2\pi - \frac{3h^2\pi}{4}\right]$$

$\frac{d^2V}{dh^2} = 0 - \frac{3(2h)\pi}{3} = -2h\pi$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dh^2}\right)_{h=10} = -2h\pi = -2(10)\pi = -20\pi$$

As $\left(\frac{d^2V}{dh^2}\right)_{h=10} = -20\pi < 0$, so the function V is maximum at h= 10

Substituting h in equation (1)

$$r^2 = \frac{1}{4}[4(5\sqrt{3})^2 - (10)^2]$$

$$r^2 = \frac{1}{4}[4(25 \times 3) - 100]$$

$$r^2 = \frac{300 - 100}{4} = \frac{200}{4} = 50$$

As V is maximum, substituting h and r in the volume formula:

$V = \pi (50) (10)$

$V = 500\pi \text{ cm}^3$

Therefore when the volume of a inscribed cylinder is maximum and is equal $500\pi \text{ cm}^3$

Question: 30

A square tank of

Solution:

Given,

- Capacity of the square tank is 250 cubic metres.
- Cost of the land per square meter Rs.50.
- Cost of digging the whole tank is Rs. $(400 \times h^2)$.
- Where h is the depth of the tank.

Let us consider,

- Side of the tank is x metres.
- Cost of the digging is; $C = 50x^2 + 400h^2$ -----(1)
- Volume of the tank is; $V = x^2h$; $250 = x^2h$

$h = \frac{250}{x^2}$ ----- (2)

Substituting (2) in (1),

$$C = 50x^2 + 400\left(\frac{250}{x^2}\right)^2$$

$C = 50x^2 + \frac{400 \times 62500}{x^4}$ ----- (3)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function C(x) has a maximum/minimum at a point c then C′(c) = 0.

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[50x^2 + \frac{400 \times 62500}{x^4}\right]$$

$$\frac{dC}{dx} = 50\,(2x) + \frac{25000000\,(-4)}{x^5}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dC}{dx} = 100x - \frac{10^8}{x^5}$ ....(4)

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = 100x - \frac{10^8}{x^5} = 0$$

$x^6 = 10^6$

$x = 10$

Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = \frac{d}{dx}\left[100x - \frac{10^8}{x^5}\right]$$

$\frac{d^2C}{dx^2} = 100 - \frac{10^8(-5)}{x^6} = 100 + \frac{10^8(5)}{x^6}$ ----- (5)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=10} = 100 + \frac{10^8\,(5)}{(10)^6} = 100 + 500 = 600$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=10} = 600 > 0$, so the function C is minimum at x=10

Substituting x in equation (2)

$$h = \frac{250}{(10)^2} = \frac{250}{100} = \frac{5}{2}$$

h = 2.5 m

Therefore when the cost for the digging is minimum, when x = 10m and h = 2.5m

Question: 31

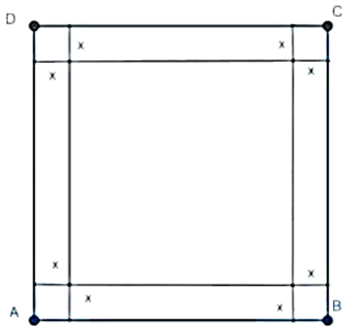
A square piece of

Solution:

Given,

• Side of the square piece is 18 cms.

• the volume of the formed box is maximum.



Let us consider,

- 'x' be the length and breadth of the piece cut from each vertex of the piece.
- Side of the box now will be (18-2x)
- The height of the new formed box will also be 'x'.

Let the volume of the newly formed box is :

$V = (18-2x)^2 \times (x)$

$V = (324+ 4x^2 - 72x)\, x$

$V = 4x^3 - 72x^2 + 324x$ ------(1)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function V(x) has a maximum/minimum at a point c then V'(c) = 0.

Differentiating the equation (1) with respect to x:

$$\frac{dV}{dx} = \frac{d}{dx}[4x^3 - 72x^2 + 324x]$$

$$\frac{dV}{dx} = 12x^2 - 144x + 324 \text{ ....(2)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dV}{dx} = 12x^2 - 144x + 324 = 0$$

$x^2 - 12x + 27 = 0$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(27)}}{2(1)} = \frac{12 \pm \sqrt{144 - 108}}{2} = \frac{12 \pm \sqrt{36}}{2}$$

$$x = = \frac{12 \pm 6}{2}$$

x = 9 or x =3

x= 2

[as x = 9 is not a possibility, because 18-2x = 18-18= 0]

Now to check if this critical point will determine the maximum area of the box, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with x:

$$\frac{d^2V}{dx^2} = \frac{d}{dx}[12x^2 - 144x + 324]$$

$$\frac{d^2V}{dx^2} = 24x - 144 \text{ ----- (4)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3) - 144 = 72 - 144 = -72$$

As $\left(\frac{d^2V}{dx^2}\right)_{x=3} = -72 < 0$, so the function V is maximum at x = 3cm

Now substituting x = 3 in 18 – 2x, the side of the considered box:

Side = 18-2x = 18 - 2(3) = 18-6= 12cm

Therefore side of wanted box is 12cms and height of the box is 3cms.

Now, the volume of the box is

$V = (12)^2 \times 3 = 144 \times 3 = 432\text{cm}^3$

Hence maximum volume of the box formed by cutting the given 18cms sheet is 432cm^3 with 12cms side and 3cms height.

Question: 32

An open tank with

Solution:

Given,

- The tank is square base open tank.
- The cost of the construction to be least.

Let us consider,

- Side of the tank is x metres.
- Height of the tank be 'h' metres.
- Volume of the tank is; $V = x^2h$
- Surface Area of the tank is $S = x^2 + 4xh$
- Let Rs.P is the price per square.

Volume of the tank,

$h = \frac{V}{x^2}$ ---- (1)

Cost of the construction be:

$C = (x^2 + 4xh)P$ -----(2)

Substituting (1) in (2),

$$C = \left[x^2 + 4x\frac{V}{x^2}\right]P$$

$C = [x^2 + \frac{4V}{x}]P$ ----- (3)

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. This is because if the function C(x) has a maximum/minimum at a point c then C'(c) = 0.

Differentiating the equation (3) with respect to x:

$$\frac{dC}{dx} = \frac{d}{dx}\left[x^2 + \frac{4V}{x}\right]P$$

$$\frac{dC}{dx} = \left[(2x) + \frac{4V(-1)}{x^2}\right]P$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

$$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P \text{ ....(4)}$$

To find the critical point, we need to equate equation (4) to zero.

$$\frac{dC}{dx} = \left[2x - \frac{4V}{x^2}\right]P = 0$$

$x^3 = 2V$

Now to check if this critical point will determine the minimum volume of the tank, we need to check with second differential which needs to be positive.

Consider differentiating the equation (4) with x:

$$\frac{d^2C}{dx^2} = P\frac{d}{dx}\left[2x - \frac{4V}{x^2}\right]$$

$$\frac{d^2C}{dx^2} = \left[2 - \frac{4V(-2)}{x^3}\right]P = \left[2 + \frac{8V}{x^3}\right]P \text{ ----- (5)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$]

Now let us find the value of

$$\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = \left[2 + \frac{8V}{2V}\right]P = [2+4]P = 6P$$

As $\left(\frac{d^2C}{dx^2}\right)_{x=(2V)^{\frac{1}{3}}} = 6P > 0$, so the function C is minimum at $x = \sqrt[3]{2V}$

Substituting x in equation (2)

$$h = \frac{V}{(2V)^{\frac{2}{3}}} = \frac{V\sqrt[3]{(2V)}}{2V} = \frac{1}{2}\sqrt[3]{2V}$$

$$h = \frac{1}{2}\sqrt[3]{2V}$$

Therefore when the cost for the digging is minimum, when $x = \sqrt[3]{2V}$ and $h = \frac{1}{2}\sqrt[3]{2V}$

Question: 33

A wire of length

Solution:

Given,

- Length of the wire is 36 cm.
- The wire is cut into 2 pieces.
- One piece is made to a square.
- Another piece made into a equilateral triangle.

Let us consider,

- The perimeter of the square is x.
- The perimeter of the equilateral triangle is (36-x).
- Side of the square is $\frac{x}{4}$

- Side of the triangle is $\frac{(36-x)}{3}$

Let the Sum of the Area of the square and triangle is

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{36-x}{3}\right)^2$$

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(12 - \frac{x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)$$

$$A = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right) \text{ --- (1)}$$

For finding the maximum/ minimum of given function, we can find it by differentiating it with x and then equating it to zero. <u>This is because if the function A(x) has a maximum/minimum at a point c then A'(c) = 0.</u>

Differentiating the equation (1) with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)\right]$$

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(0 + \frac{2x}{9} - 8\right)$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right) \text{ ----- (2)}$$

To find the critical point, we need to equate equation (2) to zero.

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right) = 0$$

$$\frac{2x}{16} = \frac{\sqrt{3}}{4}\left(8 - \frac{2x}{9}\right)$$

$$\frac{2x}{16} = 2\sqrt{3} - \frac{\sqrt{3}x}{18}$$

$$\frac{2x}{16} + \frac{\sqrt{3}x}{18} = 2\sqrt{3}$$

$$x\left(\frac{2(9) + \sqrt{3}(8)}{144}\right) = 2\sqrt{3}$$

$$x\left(\frac{18 + 8\sqrt{3}}{144}\right) = 2\sqrt{3}$$

$$x = 2\sqrt{3}\left(\frac{144}{18 + 8\sqrt{3}}\right) = \frac{144\sqrt{3}}{(9 + 4\sqrt{3})}$$

<u>Now to check if this critical point will determine the minimum area, we need to check with second differential which needs to be positive.</u>

Consider differentiating the equation (3) with x:

$$\frac{d^2A}{dx^2} = \frac{d}{dx}\left[\frac{2x}{16} + \frac{\sqrt{3}}{4}\left(\frac{2x}{9} - 8\right)\right]$$

$$\frac{d^2A}{dx^2} = \frac{1}{8} + \frac{\sqrt{3}}{4}\left(\frac{2}{9}\right) = \frac{9 + 4\sqrt{3}}{72} \text{ ----- (4)}$$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$$\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9+4\sqrt{3}}{72}$$

As $\left(\frac{d^2A}{dx^2}\right)_{x=\frac{144\sqrt{3}}{(9+4\sqrt{3})}} = \frac{9+4\sqrt{3}}{72} > 0$, so the function A is minimum at

$$x = \frac{144\sqrt{3}}{(9+4\sqrt{3})}$$

Now, the length of each piece is $x = \frac{144\sqrt{3}}{(9+4\sqrt{3})}$ cm and $36 - x = 36 - \frac{144\sqrt{3}}{(9+4\sqrt{3})} = \frac{324}{(9+4\sqrt{3})}$ cm

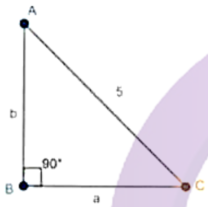
Question: 34

Find the largest

Solution:

Given,

- The triangle is right angled triangle.
- Hypotenuse is 5cm.



Let us consider,

- The base of the triangle is 'a'.
- The adjacent side is 'b'.

Now $AC^2 = AB^2 + BC^2$

As AC = 5, AB = b and BC = a

$25 = a^2 + b^2$

$b^2 = 25 - a^2$ ---- (1)

Now, the area of the triangle is

$$A = \frac{1}{2}ab$$

Squaring on both sides

$$A^2 = \frac{1}{4}a^2b^2$$

Substituting (1) in the area formula

$Z = A^2 = \frac{1}{4}a^2(25 - a^2)$ ----- (2)

For finding the maximum/ minimum of given function, we can find it by differentiating it with a and then equating it to zero. This is because if the function Z (x) has a maximum/minimum at a point c then Z'(c) = 0.

Differentiating the equation (2) with respect to a:

$$\frac{dZ}{da} = \frac{d}{da}\left[\frac{1}{4}a^2(25 - a^2)\right]$$

$\frac{dZ}{da} = \frac{1}{4}\,[\,25\,(2a) - 4a^3]$

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

$\frac{dZ}{da} = \frac{25a}{2} - a^3$ ----- (3)

To find the critical point, we need to equate equation (3) to zero.

$\frac{dZ}{da} = \frac{25a}{2} - a^3 = 0$

$a\left(\frac{25}{2} - a^2\right) = 0$

a=0 (or) $a = \frac{5}{\sqrt{2}}$

$a = \frac{5}{\sqrt{2}}$

[as a cannot be zero]

Now to check if this critical point will determine the maximum area, we need to check with second differential which needs to be negative.

Consider differentiating the equation (3) with a:

$\frac{d^2Z}{da^2} = \frac{d}{da}\left[\frac{25a}{2} - a^3\right]$

$\frac{d^2Z}{da^2} = \frac{25}{2} - 3a^2$ ----- (4)

[Since $\frac{d}{dx}(x^n) = nx^{n-1}$]

Now let us find the value of

$\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = \frac{25}{2} - 3\left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{2} - \frac{(3)25}{2} = -25$

As $\left(\frac{d^2Z}{da^2}\right)_{a=\frac{5}{\sqrt{2}}} = -25 < 0$, so the function A is maximum at $a = \frac{5}{\sqrt{2}}$

Substituting value of A in (1)

$b^2 = 25 - \frac{25}{2} = \frac{25}{2}$

$b = \frac{5}{\sqrt{2}}$

Now the maximum area is

$A = \frac{1}{2}\left(\frac{5}{\sqrt{2}}\right)\left(\frac{5}{\sqrt{2}}\right) = \frac{25}{4}$

$\therefore A = \frac{25}{4}\ cm^2$

Exercise : 11G

Question: 1

Show that t

Solution:

Domain of the function is R

Finding derivative f'(x)=5

Which is greater than 0

Mean strictly increasing in its domain i.e R

Question: 2

Show the fu

Solution:

Domain of the function is R

Finding derivative f'(x)=-2

Which is less than 0

Means strictly decreasing in its domain i.e R

Question: 3

Prove that

Solution:

Domain of the function is R

Finding derivative i.e f'(x)=a

As given in question it is given that a>0

Derivative>0

Means strictly increasing in its domain i.e R

Question: 4

Prove that

Solution:

Domain of the function is R

finding derivative i.e $f'(x)=2e^x$

As we know e^x is strictly increasing its domain

f'(x)>0

hence f(x) is strictly increasing in its domain

Question: 5

Solution:

Domain of function is **R.**

f'(x)=2x

for x>0 f'(x)>0 i.e. increasing

for x<0 f'(x)<0 i.e. decreasing

hence it is neither increasing nor decreasing in R

Question: 6

Solution:

For x>0

Modulus will open with + sign

f(x)=+x

⇒ f'(x)=+1 which is <0

for x<0

Modulus will open with -ve sign

f(x)=-x => f'(x)=-1 which is >0

hence f(x) is increasing in x>0 and decreasing in x<0

Question: 7

Prove that

Solution:

f(x)=ln(x)

$f'(x) = \frac{1}{x}$

for x<0

f'(x)=-ve →increasing

for x>0

f'(x)=+ve →decreasing

f(x) in increasing when x>0 i.e x∈(0,∞)

Question: 8

Solution:

Consider $f(x)=\log_a x$

domain of f(x) is x>0

$f'(x) = \frac{1}{x}\ln(a)$

⇒ for a>1, ln(a)>0,

hence f'(x) >0 which means strictly increasing.

⇒ for 0<a<1, ln(a)<0,

hence f'(x)<0 which means strictly decreasing.

Question: 9

Solution:

Consider $f(x)=3^x$

The domain of f(x) is R.

$f'(x)=3^x\ln(3)$

3^x is always greater than 0 and ln(3) is also + ve.

Overall f'(x) is >0 means strictly increasing in its domain i.e. R.

Question: 10

Solution:

Consider $f(x)=x^3-15x^2+75x-50$

Domain of the function is R.

$f'(x)=3x^2-30x+75$

$=3(x^2-10+25)$

$=3(x-5)\ (x-5)$

$=3(x-5)^2$

$f'(x)=0$ for $x=5$

for $x<5$

$f'(x)>5$

and

for $x>5$

$f'(x)>5$

we can see throughout R the derivative is +ve but at x=5 it is 0 so it is increasing.

Question: 11

Solution:

$$f(x) = \left(x - \frac{1}{x}\right)$$

domain of function is R-{0}

$$f'(x) = 1 + \frac{1}{x^2}$$

$f'(x)\ \forall\ x \in R$ is greater than 0.

Question: 12

Solution:

$$f(x) = \frac{1}{x} + 5$$

domain of function is R-{0}

$$f'(x) = -\frac{1}{x^2}$$

for all x, $f'(x)<0$

Hence function is decreasing.

Question: 13

Solution:

Consider $f(x) = \frac{1}{(1+x^2)}$

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

for $x \geq 0$,

$f'(x)$ is -ve.

hence function is decreasing for $x \le 0$

Question: 14

Solution:

$f(x)=x^3+x^{-3}$

$f'(x)=3x^2-3x^{-4}$

$=3(x^2-1/x^4)$

$$= 3\left(\frac{x^3-1}{x^2}.\frac{x^3+1}{x^2}\right)$$

$$= \frac{3(x-1)(x^2+x+1)(x+1)(x^2-x+1)}{x^4}$$

Root of $f'(x)=1$ and -1

Here we can clearly see that $f'(x)$ is decreasing in $[-1,1]$

So, $f(x)$ is decreasing in interval $[-1,1]$

Question: 15

Solution:

Consider $f(x) = \frac{x}{\sin x}$,

$$f'(x) = \frac{\sin x - x.\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\cos x(\tan x - x)}{\sin^2 x}$$

in $\left]0,\frac{\pi}{2}\right[$ cos>0,

tan x-x>0,

$\sin^2 x>0$

hence $f'(x)>0$,

so, function is increasing in the given interval.

Question: 16

Solution:

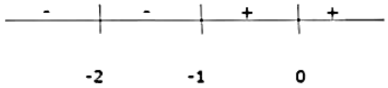
Consider

$f(x) = \log(1+x) - \frac{2x}{(x+2)}$,

$$f'(x) = \frac{1}{1+x} - \frac{4}{(x+2)^2}$$

$$= \frac{(x+2)^2 - 4(x+1)}{(x+1)(x+2)^2}$$

$$= \frac{x^2}{(x+1)(x+2)^2}$$



Clearly we can see that f'(x)>o for x>-1.

Hence function is increasing for all x>-1

Question: 17

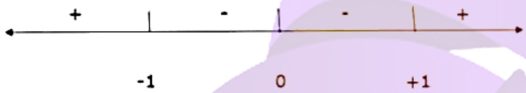
Solution:

Consider $f(x) = \left(x + \frac{1}{x}\right)$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f'(x) = \frac{x^2 - 1}{x^2}$$

$$= \frac{x - 1 . x + 1}{x^2}$$



We can see f'(x) <0 in [-1,1]

i.e. f(x) is decreasing in this interval.

We can see f'(x) >0 in (-∞, -1) ∪(1, ∞)

i.e. f(x) is increasing in this interval.

Question: 18

Solution:

Consider $f(x) = \frac{(x-2)}{(x+1)}$

$$f'(x) = \frac{3}{(x + 1)^2}$$

f'(x) at x=-1 is not defined

and for all x ∈ R- {-1}

f'(x)>0

hence f(x) is increasing.

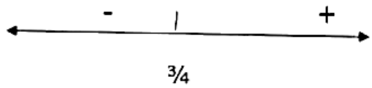
Question: 19

Solution:

$f(x)=(2x^2-3x)$

f'(x)=4x-3

f'(x)=0 at x=3/4



Clearly we can see that function is increasing for x∈[3/4, ∞) and is decreasing for x∈(-∞,3/4)

Question: 20

Solution:

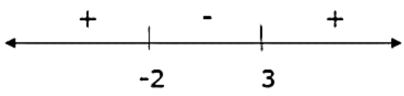
$f(x)=2x^3-3x^2-36x+7$

$f'(x)=6x^2-6x-36$

$f'(x)=6(x^2-x-6)$

$f'(x)=6(x-3)(x+2)$

$f'(x)$ is 0 at $x=3$ and $x=-2$



$F'(x)>0$ for $x \in (-\infty, -2] \cup [3, \infty)$

hence in this interval function is increasing.

$F'(x)<0$ for $x \in (-2, 3)$

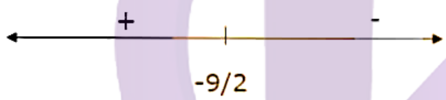
hence in this interval function is decreasing.

Question: 21

Solution:

$f(x)=6-9x-x^2$

$f'(x)=-(2x+9)$



We can see that f(x) is increasing for $x \in \left(-\infty, -\frac{9}{2}\right]$ and decreasing in $x \in \left(-\frac{9}{2}, \infty\right)$

Question: 22

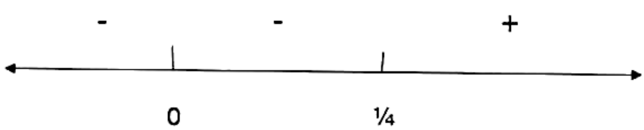
Solution:

Consider $f(x) = \left(x^4 - \frac{x^3}{3}\right)$

$f'(x)=4x^3-x^2$

$=x^2(4x-1)$

$F'(x)=0$ for $x=0$ and $x=1/4$



Function f(x) is decreasing for $x \in (-\infty, 1/4]$ and increasing in $x \in (1/4, \infty)$

Question: 23

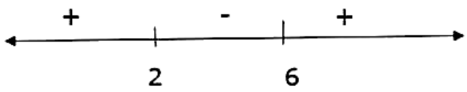
Solution:

$f(x)=x^3-12x^2+36x+17$

$f'(x)=3x^2-24x+36$

$f'(x)=3(x^2-8x+12)$

$=3(x-6)(x-2)$



Function f(x) is decreasing for $x \in [2,6]$ and increasing in $x \in (-\infty,2) \cup (6, \infty)$

Question: 24

Solution:

$f(x)=x^3-6x^2+9x+10$

$f'(x)=3x^2-12x+9$

$f'(x)=3(x^2-4x+3)$

$=3(x-3)(x-1)$

Function f(x) is decreasing for $x \in [1,3]$ and increasing in $x \in (-\infty,1) \cup (3, \infty)$

Question: 25

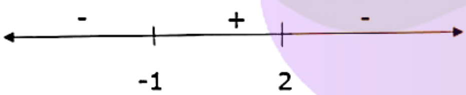
Solution:

$f(x)=-2x^3+3x^2+12x+6$

$f'(x)=-6x^2+6x+12$

$f'(x)=-6(x^2-x-2)$

$=-6(x-2)(x+1)$



Function f(x) is increasing for $x \in [-1,2]$ and decreasing in $x \in (-\infty,-1) \cup (2, \infty)$

Question: 26

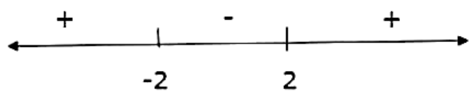
Solution:

$f(x)=2x^3-24x+5$

$f'(x)=6x^2-24$

$f'(x)=6(x^2-4)$

$=6(x-2)(x+2)$



Function f(x) is decreasing for $x \in [-2,2]$ and increasing in $x \in (-\infty,-2) \cup (2, \infty)$

Question: 27

Solution:

$f(x)=(x-1)(x-2)^2=x2-4x+4 * x-1= x3-4x2+4x-x2+4x-4$

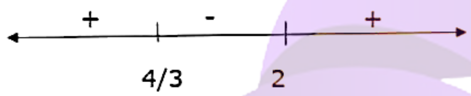
$f(x)=x^3-5x^2+8x-4$

$f'(x)=3x^2-10x+8$

$f'(x)=3x^2-6x-4x+8$

$=3x(x-2)-4(x-2)$

$=(3x-4)(x-2)$



Function f(x) is decreasing for $x \in [4/3,2]$ and increasing in $x \in (-\infty,4/3) \cup (2, \infty)$

Question: 28

Solution:

$f(x)=x^4-4x^3+4x^2+15$

$f'(x)=4x^3-12x^2+8x$

$= 4x(x^2-3x+2)$

$=4x(x-1)(x-2)$

Function f(x) is decreasing for $x \in (-\infty,0] \cup [1, 2]$ and increasing in $x \in (0,1) \cup (2, \infty)$

Question: 29

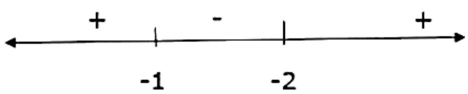
Solution:

$f(x)=2x^3+9x^2+12x+15$

$f'(x)=6x^2+18x+12$

$f'(x)=6(x^2+3x+2)$

$=6(x+2)(x+1)$



Function f(x) is decreasing for $x \in [-1,-2]$ and increasing in $x \in (-\infty,-1) \cup (-2, \infty)$

Question: 30

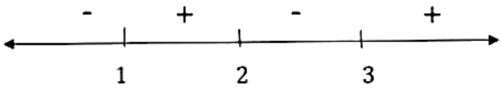
Solution:

$f(x)=x^4-8x^3+22x^2-24x+21$

$f'(x)=4x^3-24x^2+44x-24$

$= 4(x^3 -6x^2+11x-6)$

$=4(x-3)(x-1)(x-2)$



Function f(x) is decreasing for $x \in (-\infty,1] \cup [2, 3]$ and increasing in $x \in (1,2) \cup (3, \infty)$

Question: 31

Solution:

$f(x)=3x^4-4x^3-12x^2+5$

$f'(x)=12x^3-12x^2-24x$

$=12x(x^2 -x-2)$

$=12(x)(x+1)(x-2)$

Function f(x) is decreasing for $x \in (-\infty,-1] \cup [0, 2]$ and increasing in $x \in (-1,0) \cup (2, \infty)$

Question: 32

Find the interval

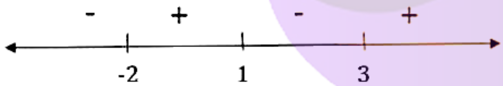
Solution:

$f'(x) = \frac{12x^3}{10} - \frac{12x^2}{5} - 6x + \frac{36}{5}$

$f'(x)=(12x^3-24x^2-60x+72)/10$

$=1.2(x^3-2x^2-5x+6)$

$=1.2(x-1)(x-3)(x+2)$



Function f(x) is decreasing for $x \in (-\infty,-2] \cup [1, 3]$ and increasing in $x \in (-2,1) \cup (3, \infty)$

Exercise : 11H

Question: 1

Solution:

i. $\frac{dy}{dx} = 3x^2 - 1$

$\frac{dy}{dx}$ at $(x = 2) = 11$

ii. $\frac{dy}{dx} = 4x + 3\cos x$

$\frac{dy}{dx}$ at $(x = 0) = 3$

iii. $\frac{dy}{dx} = 2(\sin 2x + \cot x + 2)(2\cos 2x - \operatorname{cosec}^2 x)$

$\frac{dy}{dx}$ at $\left(x = \frac{\pi}{2}\right) = 2(0 + 0 + 2)(-2 - 1) = -12$

Question: 2

Solution:

$m : \frac{dy}{dx} = 3x^2 - 2$

m at (1, 6) = 1

Tangent : $y - b = m(x - a)$

$y - 6 = 1(x - 1)$

$x - y + 5 = 0$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - 6 = -1(x - 1)$

$x + y - 7 = 0$

Question: 3

Solution:

$m : 2y\frac{dy}{dx} = 4a$

m at $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = m$

Tangent : $y - b = m(x - a)$

$y - \frac{2a}{m} = m\left(x - \frac{a}{m^2}\right)$

$m^2x - my + a = 0$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - \frac{2a}{m} = \frac{-1}{m}\left(x - \frac{a}{m^2}\right)$

$m^2x + m^3y - 2am^2 - a = 0$

Question: 4

Solution:

$m : \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

m at $(a\cos\theta, b\sin\theta) = \frac{-b\cos\theta}{a\sin\theta}$

Tangent : $y - b = m(x - a)$

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$bx\cos\theta + ay\sin\theta = ab$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$

Question: 5

Solution:

$m : \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$m \text{ at } (a\sec\theta, b\tan\theta) = \frac{b\sec\theta}{a\tan\theta}$

Tangent : $y - b = m(x - a)$

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$bx\sec\theta - ay\tan\theta = ab$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - b\sin\theta = \frac{-a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$by\,\mathrm{cosec}\,\theta + ax\sec\theta = (a^2 + b^2)$

Question: 6

Solution:

$m : \frac{dy}{dx} = 3x^2$

$m \text{ at } (1, 1) = 3$

Tangent : $y - b = m(x - a)$

$y - 1 = 3(x - 1)$

$y = 3x - 2$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - 1 = \frac{-1}{3}(x - 1)$

$x + 3y = 4$

Question: 7

Solution:

$m : 2y\frac{dy}{dx} = 4a$

$m \text{ at } (at^2, 2at) = 1/t$

Tangent : $y - b = m(x - a)$

$y - 2at = \frac{1}{t}(x - at^2)$

$x - ty + at^2 = 0$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - 2at = -t(x - at^2)$

$tx + y = at^3 + 2at$

Question: 8

Solution:

$m : \frac{dy}{dx} = 2\cot x\,(-\mathrm{cosec}^2 x) + 2\,\mathrm{cosec}^2 x$

m at $(x = \pi/4) = 2(-2) + 2(2) = 0$

Tangent : $y - b = m(x - a)$

$y - 1 = 0(x - \pi/4)$

$y = 1$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - 1 = \frac{-1}{0}\left(x - \frac{\pi}{4}\right)$

$x = \pi/4$

Question: 9

Solution:

$m : 32x + 18y\frac{dy}{dx} = 0$

m at $(2, y_1) = \frac{-32}{9y_1}$

$16(2)^2 + 9(y_1)^2 = 144$

$y_1 = \frac{4\sqrt{5}}{3}$

Tangent : $y - b = m(x - a)$

$y - \frac{4\sqrt{5}}{3} = \frac{-32}{9\frac{4\sqrt{5}}{3}}(x - 2)$

$8x + 3\sqrt{5}y - 36 = 0$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - \frac{4\sqrt{5}}{3} = \frac{9\frac{4\sqrt{5}}{3}}{32}(x - 2)$

$9\sqrt{5x} - 24y + 14\sqrt{5} = 0$

Question: 10

Solution:

$m : \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$

m at (x = 1) = 2

y at (x = 1) = $(1)^4 - 6(1)^3 + 13(1)^2 - 10(1) + 5 = 3$

Tangent : $y - b = m(x - a)$

$y - 3 = 2(x - 1)$

$2x - y + 1 = 0$

Normal : $y - b = \frac{-1}{m}(x - a)$

$y - 3 = \frac{-1}{2}(x - 1)$

$x + 2y - 7 = 0$

Question: 11

Solution:

m : $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

m at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right) = -1$

$y - b = m(x - a)$

$y - \frac{a^2}{4} = -1\left(x - \frac{a^2}{4}\right)$

$2(x + y) = a^2$

Question: 12

Solution:

m at $(x_1, y_1) = \frac{b^2x_1}{a^2y_1}$

At (x_1, y_1) : $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \Rightarrow b^2x_1^2 - a^2y_1^2 = a^2b^2$

$y - b = m(x - a)$

$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$

$a^2y_1y - a^2y_1^2 = b^2x_1x - b^2x_1^2$

$b^2x_1x - a^2y_1y = a^2b^2$

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

Question: 13

Solution:

m : $\frac{dy}{dx} = 4\sec^3x(\tan x \sec x) - 4\tan^3x(\sec^2x)$

m at $\left(x = \frac{\pi}{3}\right) = 4(2)^3(\sqrt{3} \times 2) - 4(\sqrt{3})^3(2)^2 = 16\sqrt{3}$

At $x = \pi/3$, $y = 7$

y – b = m(x – a)

$$y - 7 = 16\sqrt{3}\left(x - \frac{\pi}{3}\right)$$

$$3y - 48\sqrt{3}x + 16\sqrt{3}\pi - 21 = 0$$

Question: 14

Solution:

$$m : \frac{dy}{dx} = 2(\sin 2x + \cot x + 2)(2\cos 2x - \operatorname{cosec}^2 x)$$

$$\frac{dy}{dx} \text{ at } \left(x = \frac{\pi}{2}\right) = 2(0 + 0 + 2)(-2 - 1) = -12$$

At $x = \pi/2$, $y = 4$

$$y - b = \frac{-1}{m}(x - a)$$

$$y - 4 = \frac{1}{12}\left(x - \frac{\pi}{2}\right)$$

$24y - 2x + \pi - 96 = 0$

Question: 15

Solution:

$$m : \frac{dy}{dx} = 6x^2$$

m at (x = 2) = 24

m at (x = -2) = 24

We know that if the slope of curve at two different point is equal then straight lines are parallel at that points.

Question: 16

Solution:

We know that if two straight lines are parallel then their slope are equal. So, slope of required tangent is also equal to 4.

$$m : \frac{dy}{dx} = \frac{-2x}{3} = 4$$

x = -6 and y = -11

y – b = m(x – a)

y – (-11) = 4(x – (-6))

4x – y + 13 = 0

Question: 17

Solution:

If the tangent is parallel to y-axis it means that it's slope is not defined or 1/0.

$m: 2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-(2x-2)}{(2y-4)} = \frac{1}{0}$

$2y - 4 = 0 \Rightarrow y = 2$

$x^2 + (2)^2 - 2x - 4(2) + 1 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow x = 3$ and $x = -1$

So, the requied points are (-1, 2) and (3, 2).

Question: 18

Solution:

If the tangent is parallel to x-axis it means that it's slope is 0

$m: 2x + 2y\frac{dy}{dx} - 2 = 0$

$2x + 2y(0) - 2 = 0$

$x = 1$

$(1)^2 + y^2 - 2(1) - 3 = 0$

$\Rightarrow y^2 = 4 \Rightarrow y = 2$ and $y = -2$

So, the requied points are (1, 2) and (1, -2).

Question: 19

Solution:

We know that if the slope of two tangent of a curve are satisfies a relation $m_1m_2 = -1$, then tangents are at right angles

$m: \frac{dy}{dx} = 2x - 5$

m_1 at $(2, 0) = -1$

m_2 at $(3, 0) = 1$

$m_1m_2 = (-1)(1) = -1$

So, we can say that tangent at (2, 0) and (3, 0) are at right angles.

Question: 20

Solution:

If tangent is pass through origin it means that equation of tangent is $y = mx$

Let us suppose that tangent is made at point (x_1, y_1)

$y_1 = x_1^2 + 3x_1 + 4$ ...(1)

$m: \frac{dy}{dx} = 2x + 3$

m at $(x_1, y_1) = 2x_1 + 3$

Equation of tangent : $y_1 = (2x_1 + 3)x_1$ ...(2)

On compairing eq(1) and eq(2)

$x_1^2 + 3x_1 + 4 = (2x_1 + 3)x_1$

$x_1^2 - 4 = 0 \Rightarrow x_1 = 2$ and -2

At $x_1 = 2$, $y_1 = 14$

At $x_1 = -2$, $y_1 = 2$

So, required points are (2, 14) and (-2, 2)

Question: 21

Solution:

Slope of $y = x - 11$ is equal to 1

$m : \frac{dy}{dx} = 3x^2 - 11$

$3x^2 - 11 = 1 \Rightarrow x = 2$ and -2

At x = 2

From the equation of curve, $y = (2)^3 - 11(2) + 5 = -9$

From the equation of tangent, $y = 2 - 11 = -9$

At x = -2

From the equation of curve, $y = (-2)^3 - 11(-2) + 5 = 19$

From the equation of tangent, $y = -2 - 11 = -13$

So, the final answer is (2, -9) because at x = -2, y is come different from the equation of curve and tangent which is not possible.

Question: 22

Solution:

If tangent is parallel to the line $x + 3y = 4$ then it's slope is -1/3.

$m : 4x + 6y\frac{dy}{dx} = 0$

$$m = \frac{-2x}{3y} = \frac{-2x}{3\sqrt{\frac{14 - 2x^2}{3}}} = \frac{-1}{3}$$

$$2x = \sqrt{\frac{14 - 2x^2}{3}}$$

$$4x^2 = \frac{14 - 2x^2}{3}$$

x = 1 and -1

At x = 1, y = 2 and y = -2 (not possible)

At x = -1, y = -2 and y = 2 (not possible)

$y - b = m(x - a)$

At (1, 2)

$y-2=\frac{-1}{3}(x-1)$

3y + x = 7

At (-1, -2)

$y-(-2)=\frac{-1}{3}(x-(-1))$

3y + x = -7

Question: 23

Solution:

.: If tangent is perpendicular to the line x – 2y + 1 = 0 then it's -1/m is -2.

$m:\ 2x+2\frac{dy}{dx}=0$

m = -x = 1/2

x = -1/2

At x = -1/2, y = 31/8

$y-b=\frac{-1}{m}(x-a)$

At (-1/2, 31/8)

$y-\frac{31}{8}=\frac{-1}{\frac{1}{2}}\left(x-\left(-\frac{1}{2}\right)\right)$

16x + 8y – 23 = 0

Question: 24

Solution:

We know that if tangent is parallel to x-axis then it's slope is equal to 0.

$m:\ \frac{dy}{dx}=4x-6$

4x – 6 = 0 ⇒ x = 3/2

At x = 3/2 , y = -17/2

So, the required points are $\left(\frac{3}{2},\frac{-17}{2}\right)$.

Question: 25

Solution:

If the tangent is parallel to chord joining the points (3, 0) and (4, 1) then slope of tangent is equal to slope of chord.

$m=\frac{1-0}{4-3}=1$

$m:\ \frac{dy}{dx}=2(x-3)$

2(x – 3) = 1 ⇒ x = 7/2

At x = 7/2, y = 1/4

So, the required points are $\left(\frac{7}{2}, \frac{1}{4}\right)$.

Question: 26

Solution:

If curves cut at right angle if $8k^2 = 1$ then vice versa also true. So, we have to prove that $8k^2 = 1$ if curve cut at right angles.

If curve cut at right angle then the slope of tangent at their intersecting point satisfies the relation $m_1m_2 = -1$

We have to find intersecting point of two curves.

$x = y^2$ and $xy = k$ then $y = k^{\frac{1}{3}}$ and $x = k^{\frac{2}{3}}$

$$m_1 : \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$m_1 \text{ at } \left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right) = \frac{1}{2k^{\frac{1}{3}}}$$

$$m_2 : \frac{dy}{dx} = \frac{-k}{x^2}$$

$$m_2 \text{ at } \left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right) = \frac{-k}{k^{\frac{4}{3}}} = -\frac{1}{k^{\frac{1}{3}}}$$

$$m_1m_2 = -1$$

$$\left(\frac{1}{2k^{\frac{1}{3}}}\right)\left(-\frac{1}{k^{\frac{1}{3}}}\right) = -1$$

$$k^{\frac{2}{3}} = \frac{1}{2} \Rightarrow k^2 = \frac{1}{8} \Rightarrow 8k^2 = 1$$

Question: 27

Solution:

If the two curve touch each other then the tangent at their intersecting point formed a angle of 0.

We have to find the intersecting point of these two curves.

$xy = a^2$ and $x^2 + y^2 = 2a^2$

$\Rightarrow x^2 + \left(\frac{a^2}{x}\right)^2 = 2a^2$

$\Rightarrow x^4 - 2a^2x^2 + a^4 = 0$

$\Rightarrow (x^2 - a^2) = 0$

$\Rightarrow x = +a$ and $-a$

At $x = a$, $y = a$

At $x = -a$, $y = -a$

$$m_1 : \frac{dy}{dx} = \frac{-a^2}{x^2}$$

m_1 at $(a, a) = -1$

m_1 at $(-a, -a) = -1$

$m_2: \ 2x + 2y\frac{dy}{dx} = 0$

m_2 at (a, a) = -1

m_2 at (-a, -a) = -1

At (a, a)

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$\tan\theta = \frac{(-1)-(-1)}{1+(-1)(-1)} = 0 \Rightarrow \theta = 0$

At (-a, -a)

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$\tan\theta = \frac{(-1)-(-1)}{1+(-1)(-1)} = 0 \Rightarrow \theta = 0$

So, we can say that two curves touch each other because the angle between two tangent at their intersecting point is equal to 0.

Question: 28

Solution:

If the two curve cut orthogonally then angle between their tangent at intersecting point is equal to 90^0.

We have to find their intersecting point.

$x^3 - 3xy^2 + 2 = 0$ ...(1) and $3x^2y - y^3 - 2 = 0$ ...(2)

On adding eq (1) and eq (2)

$x^3 - 3xy^2 + 2 + 3x^2y - y^3 - 2 = 0$

$x^3 - y^3 - 3xy^2 + 3x^2y = 0$

$(x - y)^3 = 0 \Rightarrow x = y$

Put x = y in eq (1)

$y^3 - 3y^3 + 2 = 0 \Rightarrow y = 1$

At y = 1, x = 1

$m_1: \ 3x^2 - 3\left(x \times 2y\frac{dy}{dx} + y^2\right) = 0$

m_1 at (1, 1) = 0

$m_2: \ 3\left(x^2\frac{dy}{dx} + 2xy\right) - 3y^2\frac{dy}{dx} = 0$

m_2 at (1, 1) = -2/0

At (1, 1)

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$\tan\theta = \frac{m_2\left(1 - \frac{m_1}{m_2}\right)}{m_2\left(\frac{1}{m_2} + m_1\right)}$

$$\tan\theta = \frac{(1-0)}{(0+0)} = \text{not defined} \Rightarrow \theta = \frac{\pi}{2}$$

So, we can say that two curve cut each other orthogonally because angle between two tangent at their intersecting point is equal to 90^0.

Question: 29

Solution:

$$m : \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin\theta}{1+\cos\theta}$$

$$m \text{ at } \left(\theta = \frac{\pi}{4}\right) = \frac{-1}{1+\sqrt{2}} = 1-\sqrt{2}$$

At $\theta = \frac{\pi}{4}, x = \left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ and $y = \left(1 + \frac{1}{\sqrt{2}}\right)$

$y - b = m(x - a)$

$$y - \left(1 + \frac{1}{\sqrt{2}}\right) = (1-\sqrt{2})\left(x - \left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)\right)$$

$$y = (1-\sqrt{2})x + \frac{(\sqrt{2}-1)\pi}{4} + 2$$

Question: 30

Solution:

$$m : \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin 2t}{3\cos 3t}$$

$$m \text{ at } \left(t = \frac{\pi}{4}\right) = \frac{2\sqrt{2}}{3}$$

At $t = \frac{\pi}{4}, x = \frac{1}{\sqrt{2}}$ and $y = 0$

$y - b = m(x - a)$

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$4x - 3\sqrt{2}y - 2\sqrt{2} = 0$$