Chapter : 22. VECTORS AND THEIR PROPERTIES

Exercise : 22

Question: 1

Solution:

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the magnitude $|\vec{a}| = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$

A. $\vec{a} = \hat{i} + 2\hat{j} + 5\hat{k}$

$\therefore |\vec{a}| = \sqrt{1^2 + 2^2 + 5^2}$

$= \sqrt{30}$ units

B. $\vec{a} = 5\hat{i} - 4\hat{j} - 3\hat{k}$

$\therefore |\vec{a}| = \sqrt{5^2 + 4^2 + 3^2}$

$= 5\sqrt{2}$ units

C. $\vec{a} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

$\therefore |\vec{a}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2}$

$= 1$ unit

D. $\vec{a} = \sqrt{2}\hat{i} + \sqrt{3}\hat{j} - \sqrt{5}\hat{k}$

$\therefore |\vec{a}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2 + (\sqrt{5})^2}$

$= \sqrt{10}$ units

Question: 2

Solution:

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the unit vector is represented as $\hat{a} = \frac{a_x\hat{i} + a_y\hat{j} + a_z\hat{k}}{\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}}$

A. $\vec{a} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$\therefore \hat{a} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{\sqrt{3^2 + 4^2 + 5^2}}$

$= \frac{3}{5\sqrt{2}}\hat{i} + \frac{4}{5\sqrt{2}}\hat{j} - \frac{5}{5\sqrt{2}}\hat{k}$

B. $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore \hat{a} = \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + 2^2 + 6^2}}$

$= \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$

C. $\vec{a} = \hat{i} + \hat{k}$

$\therefore \hat{a} = \frac{\hat{i} + \hat{k}}{\sqrt{1^2 + 1^2}}$

$= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$

D. $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$

$\therefore \hat{a} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}$

$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$

Question: 3

If $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

$\therefore \lambda\vec{a} = 2\lambda\hat{i} - 4\lambda\hat{j} + 5\lambda\hat{k}$

For a unit vector, its magnitude equals to 1.

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the magnitude $|\vec{a}| = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$

$\therefore |\lambda\vec{a}| = \sqrt{(2\lambda)^2 + (4\lambda)^2 + (5\lambda)^2} = 1$

$\Rightarrow 45\lambda^2 = 1$

$\Rightarrow \lambda^2 = \frac{1}{45} = \frac{1}{(3\sqrt{5})^2}$

$\Rightarrow \lambda = \pm\frac{1}{3\sqrt{5}}$

Question: 4

If $\vec{a} = -\hat{i} + \hat{j} - \hat{k}$

$\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\therefore \vec{a} + \vec{b}$

$= (-\hat{i} + \hat{j} - \hat{k}) + (2\hat{i} - \hat{j} + 2\hat{k})$

$= \hat{i} + \hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the unit vector is represented as $\hat{a} = \frac{a_x\hat{i} + a_y\hat{j} + a_z\hat{k}}{\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}}$

$\therefore \widehat{(\vec{a} + \vec{b})}$

$= \frac{\hat{i} + \hat{k}}{\sqrt{1^2 + 1^2}}$

$= \frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$

Question: 5

If $\vec{a} = 3\hat{i} + \hat{j} - 5\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

$\therefore \vec{a} - \vec{b}$

$= (3\hat{i} + \hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})$

$= 2\hat{i} - \hat{j} - 4\hat{k}$

CLASS24

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the unit vector is represented as $\hat{a} = \frac{a_x i + a_y}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$

$\therefore \widehat{(\vec{a} - \vec{b})}$

$= \frac{2\hat{i} - \hat{j} - 4\hat{k}}{\sqrt{2^2 + 1^2 + 4^2}}$

$= \frac{1}{\sqrt{21}}(2\hat{i} - \hat{j} - 4\hat{k})$

Question: 6

If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{b} = 2\hat{i} + 4\hat{j} + 9\hat{k}$

$\therefore \vec{a} + \vec{b}$

$= (\hat{i} + 2\hat{j} - 3\hat{k}) + (2\hat{i} + 4\hat{j} + 9\hat{k})$

$= 3\hat{i} + 6\hat{j} + 6\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the unit vector is represented as $\hat{a} = \frac{a_x i + a_y j + a_z \hat{k}}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$

$\therefore \widehat{(\vec{a} + \vec{b})}$

$= \frac{3\hat{i} + 6\hat{j} + 6\hat{k}}{\sqrt{3^2 + 6^2 + 6^2}}$

$= \pm\frac{1}{9}(3\hat{i} + 6\hat{j} + 6\hat{k})$

$= \pm\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

Question: 7

Solution:

Let λ be an arbitrary constant and the required vector is $-2\lambda\hat{i} + \lambda\hat{j} + 2\lambda\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the magnitude $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\therefore \sqrt{(2\lambda)^2 + (\lambda)^2 + (2\lambda)^2} = 9$

$\Rightarrow 3\lambda = 9$

$\Rightarrow \lambda = 3$

The required vector is $-6\hat{i} + 3\hat{j} + 6\hat{k}$

Question: 8

Solution:

Let λ be an arbitrary constant and the required vector is $5\lambda\hat{i} - \lambda\hat{j} + 2\lambda\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the magnitude $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\therefore \sqrt{(5\lambda)^2 + (\lambda)^2 + (2\lambda)^2} = 8$

$\Rightarrow \sqrt{30}\lambda = 8$

$\Rightarrow \lambda = \dfrac{8}{\sqrt{30}}$

The required vector is $\frac{8}{\sqrt{30}}\left(5\hat{\imath} - \hat{\jmath} + 2\hat{k}\right)$

Question: 9

Solution:

Let λ be an arbitrary constant and the required vector is $2\lambda\hat{\imath} - 3\lambda\hat{\jmath} + 6\lambda\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ the magnitude $|\vec{a}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$

$\therefore \sqrt{(2\lambda)^2 + (3\lambda)^2 + (6\lambda)^2} = 21$

$\Rightarrow 7\lambda = 21$

$\Rightarrow \lambda = 3$

The required vector is $\left(6\hat{\imath} - 9\hat{\jmath} + 18\hat{k}\right)$

Question: 10

If $\vec{a} = \hat{\imath} - 2\hat{\jmath}$

$\vec{b} = 2\hat{\imath} - 3\hat{\jmath}$

$\vec{c} = 2\hat{\imath} + 3\hat{k}$

$\therefore \vec{a} + \vec{b} + \vec{c}$

$= (\hat{\imath} - 2\hat{\jmath}\,) + (2\hat{\imath} - 3\hat{\jmath}\,) + (2\hat{\imath} + 3\hat{k})$

$= 5\hat{\imath} - 5\hat{\jmath} + 3\hat{k}$

Question: 11

Solution:

A = (-2, 1, 2)

B = (2, -1, 6)

$\therefore \overrightarrow{AB}$

$= \{2 - (-2)\}\hat{\imath} + \{(-1) - 1\}\hat{\jmath} + \{6 - 2\}\hat{k}$

$= 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ the unit vector is represented as $\hat{a} = \frac{a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}}{\sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}}$

$\therefore \widehat{AB}$

$= \dfrac{4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{\sqrt{4^2 + 2^2 + 4^2}}$

$= \dfrac{4}{6}\hat{\imath} - \dfrac{2}{6}\hat{\jmath} + \dfrac{4}{6}\hat{k}$

$= \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$

Question: 12

Solution:

$\vec{a} = 5\hat{i} - 3\hat{j} + 4\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the direction ratios are represented as (a_x, a_y, a_z) and the direction cosines are given by $\frac{a_x}{\sqrt{a_x^2+a_y^2+a_z^2}}, \frac{a_y}{\sqrt{a_x^2+a_y^2+a_z^2}}, \frac{a_z}{\sqrt{a_x^2+a_y^2+a_z^2}}$

The direction ratios are (5,-3, 4)

The direction cosines are $\frac{5}{\sqrt{5^2+3^2+4^2}}, \frac{-3}{\sqrt{5^2+3^2+4^2}}, \frac{4}{\sqrt{5^2+3^2+4^2}}$

$= \frac{5}{5\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}$

$= \frac{1}{\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}$

Question: 13

Solution:

A = (2,1,-2)

B = (3,5,-4)

$\therefore \overrightarrow{AB}$

$= \{3-2\}\hat{i} + \{5-1\}\hat{j} + \{(-4)-(-2)\}\hat{k}$

$= \hat{i} + 4\hat{j} - 2\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the direction ratios are represented as (a_x, a_y, a_z) and the direction cosines are given by $\frac{a_x}{\sqrt{a_x^2+a_y^2+a_z^2}}, \frac{a_y}{\sqrt{a_x^2+a_y^2+a_z^2}}, \frac{a_z}{\sqrt{a_x^2+a_y^2+a_z^2}}$

The direction ratios are (1,4, -2)

The direction cosines are $\frac{1}{\sqrt{1^2+4^2+2^2}}, \frac{4}{\sqrt{1^2+4^2+2^2}}, \frac{-2}{\sqrt{1^2+4^2+2^2}}$

$= \frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}$

Question: 14

Solution:

$A = \hat{i} + 2\hat{j} + 7\hat{k}$

$B = 2\hat{i} + 6\hat{j} + 2\hat{k}$

$C = 3\hat{i} + 10\hat{j} - 3\hat{k}$

$\therefore \overrightarrow{AB}$

$= (2\hat{i} + 6\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} + 7\hat{k})$

$= \hat{i} + 4\hat{j} - 5\hat{k}$

$\therefore \overrightarrow{BC}$

$= (3\hat{i} + 10\hat{j} - 3\hat{k}) - (2\hat{i} + 6\hat{j} + 2\hat{k})$

$= \hat{i} + 4\hat{j} - 5\hat{k}$

$\overrightarrow{AB} = \overrightarrow{BC}$

So, the points A, B and C are collinear.

Question: 15

Solution:

$A = 2\hat{i} + \hat{j} - \hat{k}$

$B = 3\hat{i} - 2\hat{j} + \hat{k}$

$C = \hat{i} + 4\hat{j} - 3\hat{k}$

$\therefore \overrightarrow{AB}$

$= (3\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j} - \hat{k})$

$= \hat{i} - 3\hat{j} + 2\hat{k}$

$\therefore \overrightarrow{BC}$

$= (\hat{i} + 4\hat{j} - 3\hat{k}) - (3\hat{i} - 2\hat{j} + \hat{k})$

$= -2\hat{i} + 6\hat{j} - 4\hat{k}$

$(-3)\overrightarrow{AB} = \overrightarrow{BC}$

So, the points A, B and C are collinear.

Question: 16

Solution:

$A = \hat{i} + 2\hat{j} + 3\hat{k}$

$B = 2\hat{i} + 3\hat{j} + \hat{k}$

$C = 3\hat{i} + \hat{j} + 2\hat{k}$

$\therefore \overrightarrow{AB}$

$= (2\hat{i} + 3\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= \hat{i} + \hat{j} - 2\hat{k}$

$\therefore \overrightarrow{BC}$

$= (3\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} + 3\hat{j} + \hat{k})$

$= \hat{i} - 2\hat{j} + \hat{k}$

$\therefore \overrightarrow{CA}$

$= (\hat{i} + 2\hat{j} + 3\hat{k}) - (3\hat{i} + \hat{j} + 2\hat{k})$

$= -2\hat{i} + \hat{j} + \hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the magnitude $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\therefore |\overrightarrow{AB}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

$\therefore |\overrightarrow{BC}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

$\therefore |\overrightarrow{CA}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$\therefore |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$

The three sides of the triangle are equal in magnitude, so the triangle is equilateral.

Question: 17

Solution:

$A = 3\hat{i} - 4\hat{j} - 4\hat{k}$

$B = 2\hat{i} - \hat{j} + \hat{k}$

$C = \hat{i} - 3\hat{j} - 5\hat{k}$

$\therefore \overrightarrow{AB}$

$= (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$

$= -\hat{i} + 3\hat{j} + 5\hat{k}$

$\therefore \overrightarrow{BC}$

$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$

$\therefore \overrightarrow{CA}$

$= (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$

$= 2\hat{i} - \hat{j} + \hat{k}$

Tip – For any 2 perpendicular vectors $\vec{a}$ & $\vec{b}$, $\vec{a}.\vec{b} = 0$

$\therefore \overrightarrow{AB}.\overrightarrow{CA}$

$= (-\hat{i} + 3\hat{j} + 5\hat{k}).(2\hat{i} - \hat{j} + \hat{k})$

$= -2 - 3 + 5$

$= 0$

The triangle is right-angled.

Question: 18

Solution:

A = (1,-1,0)

B = (4,-3,1)

C = (2,-4,5)

$\therefore \overrightarrow{AB}$

$= (4 - 1)\hat{i} + (-3 + 1)\hat{j} + (1 - 0)\hat{k}$

$= 3\hat{i} - 2\hat{j} + \hat{k}$

$\therefore \overrightarrow{BC}$

$= (2-4)\hat{i} + (-4+3)\hat{j} + (5-1)\hat{k}$

$= -2\hat{i} - \hat{j} + 4\hat{k}$

$\therefore \overrightarrow{CA}$

$= (1-2)\hat{i} + (-1+4)\hat{j} + (0-5)\hat{k}$

$= -\hat{i} + 3\hat{j} - 5\hat{k}$

Tip – For any 2 perpendicular vectors $\vec{a}$ & $\vec{b}$, $\vec{a}.\vec{b} = 0$

$\therefore \overrightarrow{AB}.\overrightarrow{BC}$

$= (3\hat{i} - 2\hat{j} + \hat{k}).(-2\hat{i} - \hat{j} + 4\hat{k})$

$= -6 + 2 + 4$

$= 0$

The triangle is right-angled.

Question: 19

Solution:

$\vec{A} = 2\hat{a} - 3\hat{b}$

$\vec{B} = 3\hat{a} - 2\hat{b}$

Formula to be used – The point dividing a line joining points a and b in a ratio m:n internally or externally is given by $\frac{mb \pm na}{m+b}$ respectively.

The position vector of the point dividing the line internally

$$= \frac{2 \times (3\hat{a} - 2\hat{b}) + 3 \times (2\hat{a} - 3\hat{b})}{2+3}$$

$$= \frac{12}{5}\hat{a} - \frac{13}{5}\hat{b}$$

The position vector of the point dividing the line externally

$$= \frac{2 \times (3\hat{a} - 2\hat{b}) - 3 \times (2\hat{a} - 3\hat{b})}{2-3}$$

$= -5\hat{b}$

Question: 20

Solution:

$\vec{A} = 2\hat{a} + \hat{b}$

$\vec{B} = \hat{a} - 3\hat{b}$

Formula to be used – The point dividing a line joining points a and b in a ratio m:n externally is given by $\frac{mb - na}{m-b}$ respectively.

The position vector of the point C dividing the line externally

$$= \frac{1 \times (\hat{a} - 3\hat{b}) - 2 \times (2\hat{a} + \hat{b})}{2-3}$$

$= 3\hat{a} + 5\hat{b}$

The midpoint of B and C may be given by

$$\frac{(\hat{a} - 3\hat{b}) + (3\hat{a} + 5\hat{b})}{2}$$

$= 2\hat{a} + \hat{b}$ i.e. point A

A is the midpoint of B and C.

Question: 21

Solution:

A = (-2,1,3)

B = (3,5,-2)

$\therefore \overrightarrow{OA} = -2\hat{i} + \hat{j} + 3\hat{k}$

$\therefore \overrightarrow{OB} = 3\hat{i} + 5\hat{j} - 2\hat{k}$

Formula to be used – The point dividing a line joining points a and b in a ratio m:n internally or externally is given by $\frac{mb \pm na}{m \pm b}$ respectively.

The position vector of the point dividing the line internally

$$= \frac{2 \times (-2\hat{i} + \hat{j} + 3\hat{k}) + 1 \times (3\hat{i} + 5\hat{j} - 2\hat{k})}{2 + 1}$$

$$= \frac{4}{3}\hat{i} + \frac{11}{3}\hat{j} - \frac{1}{3}\hat{k}$$

The position vector of the point dividing the line externally

$$= \frac{2 \times (-2\hat{i} + \hat{j} + 3\hat{k}) - 1 \times (3\hat{i} + 5\hat{j} - 2\hat{k})}{2 - 1}$$

$= 8\hat{i} + 9\hat{j} - 7\hat{k}$

Question: 22

Solution:

$\overrightarrow{OA} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$\overrightarrow{OB} = \hat{i} + 4\hat{j} - 2\hat{k}$

Formula to be used – The midpoint of a line joining points a and b is given by $\frac{a+b}{2}$.

The position vector of the midpoint

$$= \frac{(3\hat{i} + 2\hat{j} + 6\hat{k}) + (\hat{i} + 4\hat{j} - 2\hat{k})}{2}$$

$= 2\hat{i} + 3\hat{j} + 2\hat{k}$

Question: 23

If A = (1,2,-1)

Let the co-ordinates of point B be (b_1,b_2,b_3)

$\overrightarrow{AB} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\Rightarrow [(b_1 - 1)\hat{i} + (b_2 - 2)\hat{j} + (b_3 + 1)\hat{k}] = 2\hat{i} + \hat{j} - 3\hat{k}$

Comparing the respective co-efficient,

b_1-1 = 2 i.e. b_1 = 3

b_2-2 = 1 i.e. b_2 = 3

b_3+1 = -3 i.e. b_3 = -4

The required co-ordinates of B are (3,3,-4)

Question: 24

Solution:

P = (1,3,0)

Q = (4,5,6)

$\therefore \overrightarrow{PQ}$

$= (4-1)\hat{i} + (5-3)\hat{j} + (6-0)\hat{k}$

$= 3\hat{i} + 2\hat{j} + 6\hat{k}$

Tip – For any vector $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ the unit vector is represented as $\hat{a} = \frac{a_x i + a_y j + a_z \hat{k}}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$

$\therefore \widehat{PQ}$

$= \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + 2^2 + 6^2}}$

$= \frac{1}{7}(3\hat{i} + 2\hat{j} + 6\hat{k})$