Chapter : 6. DETERMINANTS

Exercise : 6A

Question: 1

Solution:

Theorem: If A be k × k matrix then $|pA|=p^k|A|$.

Given, p=4,k=2 and |A|=5.

$|4A|=4^2 \times 5$

$=16 \times 5$

$=80$

Question: 2

Solution:

Theorem: If Let A be k × k matrix then $|pA|=p^k|A|$.

Given: k=3 and p=3.

$|3A|=3^3 \times |A|$

$=27|A|$.

Comparing above with k|A| gives k=27.

Question: 3

Solution:

Theorem: If A be k × k matrix then $|pA|=p^k|A|$.

Given: p=2, k=3 and|A|=4

$|2A|=2^3 \times |A|$

$=8 \times 4$

$= 32$

Question: 4

Solution:

Theorem: A_{ij} is found by deleting i^{th} rowand j^{th} column, the determinant of left matrix is called cofactor with multiplied by $(-1)^{(i+j)}$.

Given: i=3 and j=2.

$A_{32}=(-1)^{(3+2)}(2 \times 4-6 \times 5)$

$=-1 \times (-22)$

$=22$

$a_{32}=5$

$a_{32}A_{32}= 5 \times 22$

=110

Question: 5

Solution:

Theorem: This evaluation can be done in two different ways either by taking out the common things and then calculating the determinants or simply take determinant.

I will prefer first method because with that chances of silly mistakes reduces.

Take out x+1 from second row.

$(x+1) \times \begin{vmatrix} x^2-x+1 & x-1 \\ 1 & 1 \end{vmatrix}$

$\Rightarrow (x+1) \times (x^2-x+1-(x-1))$

$\Rightarrow (x+1) \times (x^2-2x+2)$

$\Rightarrow \quad x^3-2x^2+2x+x^2-2x+2$

$\Rightarrow x^3-x^2+2$.

Question:

6

Solution:

This we can very simply go through

directly.((a+ib)(a-ib))-((-c+id)(c+id)).

$\Rightarrow (a^2+b^2)-(-c^2 -d^2)$.

$\Rightarrow a^2+b^2 + c^2+d^2$

$\because$ i × i=-1

Question:

7

Solution:

Here the determinant is compared so we need to take determinant both sides then

find x.12x+14=32-42

$\Rightarrow$ 12x=-10-14

$\Rightarrow$ 12x=-24

$\Rightarrow$ x=-2

Question:

8

Solution:

this question is having the same logic as

above.$2x^2$-40=18+14

$\Rightarrow 2x^2=72$

$\Rightarrow x^2=36$

$\Rightarrow x=\pm 6$.

Question: 9

If

Solution:

Simply by equating both sides we can get the value of

$x.2x^2+2x-2(x^2+4x+3)=-12$

$\Rightarrow -6x-6=-12$

$\Rightarrow -6x=-6$

$\Rightarrow x = 1$

Question: 10

If

Solution:

Find the determinant of A and then multiply it by 3

$|A|=2$

$3|A|=3 \times$

2

$=6$

Question: 11

Solution:

It is determinant multiplied by a scalar number 2, just find determinant of matrix and multiply it by 2.

$2 \times (35-20)$

$2 \times 15= 30$

Question: 12

Solution:

Find determinant

$\sqrt{6} \times \sqrt{24}-\sqrt{20} \times \sqrt{5}$

$\sqrt{144}-\sqrt{100}.$

$=12-10$

$=2.$

Question: 13

Solution:

After finding determinant we will get a trigonometric identity.

$2\cos^2\theta +2\sin^2\theta$

$=2$

$\because \sin^2\theta + \cos^2\theta = 1$

Question: 14

Solution:

After finding determinant we will get a trigonometric identity.

$\cos^2\alpha + \sin^2\alpha$

$= 1$

$\because \sin^2\theta + \cos^2\theta = 1$

Question: 15

Solution:

After finding determinant we will get,

$\text{Sin}60° = \frac{\sqrt{3}}{2} = \cos 30°$

$\text{Cos}60° = \frac{1}{2} = \sin 30°$

$\sin 60° \times \cos 30° + \sin 30° \times \cos 60°$

$$\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2}$$

$= \frac{3}{4} + \frac{1}{4}$

$= 1.$

Question: 16

Solution:

By directly opening this

determinant$\cos 65° \times \cos 25°$ -

$\sin 25° \times \sin 65°$

$= \cos(65°+25°)$ $\because$ $\cos A\cos B - \sin A\sin B = \cos(A+B)$

$= \cos 90°$

$= 0$

$\because$ $\cos A\cos B - \sin A\sin B = \cos(A+B)$

Question: 17

Solution:

$\cos 15°\cos 75° - \sin 75°\sin 15°$

$= \cos(15°+75°)$ $\because$ $\cos A\cos B - \sin A\sin B = \cos(A+B)$

$= \cos 90°$

$= 0$

$\because$ $\cos A\cos B - \sin A\sin B = \cos(A+B)$

Question: 18

Solution:

We know that expansion of determinant with respect to first row is

$a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$.$0(3 \times 6-5 \times 4)-2(2 \times 6-4 \times 4)+0(2 \times 5-4 \times 3)$

$= 8.$

Question: 19

Solution:

We know that $C_1 \Rightarrow C_1 - C_2$, would not change anything for the determinant.

Applying the same in above determinant, we get

$$\begin{bmatrix} 40 & 1 & 5 \\ 72 & 7 & 9 \\ 24 & 5 & 3 \end{bmatrix}$$ Now it can clearly be seen that $C_1 = 8 \times C_3$

Applying above equation we get,

$$\begin{bmatrix} 0 & 1 & 5 \\ 0 & 7 & 9 \\ 0 & 3 & 3 \end{bmatrix}$$

We know that if a row or column of a determinant is 0. Then it is singular determinant.

Question: 20

Solution:

For A to be singular matrix its determinant should be equal to 0.

0= (3-2x) × 4-(x+1) × 2

0= 12-8x-2x-2

0=10-

10xX=1.

Question: 21

Solution:

$$\begin{vmatrix} 14 & 9 \\ -8 & -7 \end{vmatrix}. = 14 \times (-7) - 9 \times (-8)$$

= -26

Question: 22

Solution:

$$\begin{vmatrix} \sqrt{3} & \sqrt{5} \\ -\sqrt{5} & 3\sqrt{3} \end{vmatrix}. = 3\sqrt{3} \times \sqrt{3} - (-\sqrt{5} \times \sqrt{5})$$

= 14.

Exercise : 6B

Question:

1 :

Solution:

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

$= \left(\frac{1}{2}\right)\begin{vmatrix}67 & 19 & 21\\78 & 26 & 28\\81 & 24 & 26\end{vmatrix}$ [R₂' = (1/2)R₂]

$= \left(\frac{1}{2}\right)\begin{vmatrix}67 & 19 & 21\\-3 & 2 & 2\\81 & 24 & 26\end{vmatrix}$ $[R_2' = R_2 - R_3]$

$= \left(\frac{1}{2}\right)\begin{vmatrix}-14 & -5 & -5\\-3 & 2 & 2\\81 & 24 & 26\end{vmatrix}$ $[R_1' = R_1 - R_3]$

$= \begin{vmatrix}-14 & -5 & -5\\-3 & 2 & 2\\81/2 & 12 & 13\end{vmatrix}$ $[R_3' = 2R_3]$

$= (-14)\{(2 \times 13) - (2 \times 12)\} - 5\{(2 \times 81/2) - (-3) \times 13\} - 5\{(-3) \times 12 - 2 \times 81/2\}$

[expanding by the first row]

$= -14 \times (26 - 24) - 5(81 + 39) - 5(-36 - 81)$

$= -14 \times 2 - 5 \times 120 - 5 \times (-117) = -28 - 600 + 585 =$ **- 43**

Question: 2

:

Solution:

$\begin{vmatrix}29 & 26 & 22\\25 & 31 & 27\\63 & 54 & 46\end{vmatrix}$

$= \begin{vmatrix}4 & -5 & -5\\25 & 31 & 27\\63 & 54 & 46\end{vmatrix}$ $[R_1' = R_1 - R_2]$

$= \left(\frac{1}{2}\right)\begin{vmatrix}4 & -5 & -5\\50 & 62 & 54\\63 & 54 & 46\end{vmatrix}$ $[R_2' = 2R_2]$

$= \left(\frac{1}{2}\right)\begin{vmatrix}4 & -5 & -5\\-13 & 8 & 8\\63 & 54 & 46\end{vmatrix}$ $[R_2' = R_2 - R_3]$

$= \begin{vmatrix}4 & -5 & -5\\-13 & 8 & 8\\63/2 & 27 & 23\end{vmatrix}$ $[R_3' = 2R_3]$

$= 4(8 \times 23 - 8 \times 27) - 5\{8 \times 63/2 - (-13) \times 23\} - 5\{(-13) \times 27 - 8 \times 63/2\}$

[expansion by first row]

= 132

Question: 3

:

Solution:

$\begin{vmatrix}102 & 18 & 36\\1 & 3 & 4\\17 & 3 & 6\end{vmatrix} = 6 \times \begin{vmatrix}17 & 18 & 6\\1 & 6 & 4\\17 & 3 & 6\end{vmatrix}$ $[R_1' = R_1/6]$

Now, for any determinant, if at least two rows are identical, then the value of the determinant becomes zero.

Here, the first and third rows are identical.

So, the value of the above determinant **d = 0**

Question: 4

:

Solution:

$$\begin{vmatrix}1^2 & 2^2 & 3^2\\ 2^2 & 3^2 & 4^2\\ 3^2 & 4^2 & 5^2\end{vmatrix} = \begin{vmatrix}1 & 4 & 9\\ 4 & 9 & 16\\ 9 & 16 & 25\end{vmatrix}$$

Expanding by first row, we get,

1(9 × 25 - 16 × 16) + 4(16 × 9 - 4 × 25) + 9(4 × 16 - 9 × 9) = - 31 + 176 - 153 = **- 8**

Question: 5

Solution:

$$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ bc & ca & ab\end{vmatrix}$$

$$= \begin{vmatrix}0 & 0 & 1\\ a-b & b-c & c\\ bc-ca & ca-ab & ab\end{vmatrix}$$ [$C_1' = C_1 - C_2$ & $C_2' = C_2 - C_3$]

$$= \begin{vmatrix}0 & 0 & 1\\ a-b & b-c & c\\ -c(a-b) & -a(b-c) & ab\end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix}0 & 0 & 1\\ 1 & 1 & c\\ -c & -a & ab\end{vmatrix}$$ [$C_1' = C_1/(a - b)$ & $C_2' = C_2/(b - c)$]

= (a - b)(b - c)[0 + 0 + 1{ - a - (- c)}] [expansion by first row]

= (a - b)(b - c)(c -a)

Question: 6

Solution:

$$\begin{vmatrix}1 & b+c & b^2+c^2\\ 1 & c+a & c^2+a^2\\ 1 & a+b & a^2+b^2\end{vmatrix}$$

$$= \begin{vmatrix}0 & b-a & b^2-a^2\\ 0 & c-b & c^2-b^2\\ 1 & a+b & a^2+b^2\end{vmatrix}$$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$$= \begin{vmatrix}0 & b-a & (b-a)(b+a)\\ 0 & c-b & (c-b)(c+b)\\ 1 & a+b & a^2+b^2\end{vmatrix}$$

$$= (b-a)(c-b)\begin{vmatrix}0 & 1 & b+a\\ 0 & 1 & c+b\\ 1 & a+b & a^2+b^2\end{vmatrix}$$ [$R_1' = R_1/(b - a)$ & $R_2' = R_2/(c - b)$]

= (b - a)(c - b)[0 + 0 + 1{(c + b) - (b + a)}] [expansion by first column]

= (a - b)(b - c)(c -a)

Question: 7

Solution:

$$\begin{vmatrix}1 & 1+p & 1+p+q\\ 2 & 3+2p & 1+3p+2q\\ 3 & 6+3p & 1+6p+3q\end{vmatrix}$$

$= \begin{vmatrix} -1 & -2-p & -2p-q \\ -1 & -3-p & -3p-q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$= \begin{vmatrix} 0 & 1 & p \\ -1 & -3-p & -3p-q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$ [$R_1' = R_1 - R_2$]

$= \left(\frac{1}{2}\right)\begin{vmatrix} 0 & 1 & p \\ -2 & -6-2p & -6p-2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$ [$R_2' = R_2*2$]

$= \left(\frac{1}{2}\right)\begin{vmatrix} 0 & 1 & p \\ 1 & p & 1+q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$ [$R_2' = R_2 + R_3$]

= (1/2)[0 + 3(1 + q) - (1 + 6p + 3q) + p(6 + 3p - 3p)] [expansion by first row]

= (1/2)(3 + 3q - 1 - 6p - 3q + 6p) = **1**

Question: 8

Solution:

$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

$= \begin{vmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a+z \end{vmatrix}$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$] $= a^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ x & y & a+z \end{vmatrix}$ [$R_1' = R_1/a$ & $R_2' = R_2/a$]

= a²[a + z - (- y) - (- x)] [expansion by first row]

= a²(a + x + y +z)

Question: 9

Solution:

$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$

$= \begin{vmatrix} x+2a & x+2a & x+2a \\ a & x & a \\ a & a & x \end{vmatrix}$ [$R_1' = R_1 + R_2 + R_3$]

$= (x+2a)\begin{vmatrix} 1 & 1 & 1 \\ a & x & a \\ a & a & x \end{vmatrix}$ [$R_1' = R_1/(x+2a)$]

$= (x+2a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-a & a-x \\ a & a & x \end{vmatrix}$ [$R_2' = R_2 - R_3$]

$= (x+2a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-a & -(x-a) \\ a & a & x \end{vmatrix}$

$= (x+2a)(x-a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ a & a & x \end{vmatrix}$ [$R_2' = R_2/(x-a)$]

= (x + 2a)(x - a)[x - (- a) + (- a - 0) + (- a)] [expansion by first row]

= (x + 2a)(x - a)(x + a - a - a) = **(x + 2a)(x -a)²**

Question: 10

Solution:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} [R_1' = R_1 + R_2 + R_3] = (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} [R_1' = R_1/(5x+4)]$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -x+4 & x-4 \\ 2x & 2x & x+4 \end{vmatrix} [R_2' = R_2 - R_3]$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(x-4) & x-4 \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)(x-4)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2x & 2x & x+4 \end{vmatrix} [R_2' = R_2/(x-4)]$$

= (5x + 4)(x - 4)[- (x + 4) - 2x + 2x - 0 + 0 - (- 2x)] [expansion by first row]

= (5x + 4)(x - 4)(- x - 4 + 2x) = **(5x + 4)(x - 4)²**

Question: 11

Solution:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5x+\lambda & 5x+\lambda & 5x+\lambda \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} [R_1' = R_1 + R_2 + R_3]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} [R_1' = R_1/(5x+\lambda)]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -x+\lambda & x-\lambda \\ 2x & 2x & x+\lambda \end{vmatrix} [R_2' = R_2 - R_3]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(x-\lambda) & x-\lambda \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= (5x+\lambda)(x-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2x & 2x & x+\lambda \end{vmatrix} [R_2' = R_2/(x-\lambda)]$$

= (5x + λ)(x - λ)[- (x + λ) - 2x + 2x - 0 + 0 - (- 2x)] [expansion by first row]

= (5x + λ)(x - λ)(- x - λ + 2x) = **(5x + λ)(x - λ)²**

Question: 12

Solution:

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ $[R_1' = R_1 - R_2 \; \& \; R_2' = R_2 - R_3]$

$= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2(a-1) & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$

$= (a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ $[R_1' = R_1/(a-1) \; \& \; R_2' = R_2/(a-1)]$

$= (a-1)^2[a+1-0-2]$ [expansion by first row]

$= (a-1)^3$

Question: 13

Solution:

$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$= \begin{vmatrix} 3(x+y) & 3(x+y) & 3(x+y) \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ $[R_1' = R_1 + R_2 + R_3]$

$= 3(x+y) \begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ $[R_1' = R_1/3(x+y)]$

$= 3(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & -2y & y \\ x+y & x+2y & x \end{vmatrix}$ $[R_2' = R_2 - R_3]$

$= 3y(x+y) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ x+y & x+2y & x \end{vmatrix}$ $[R_2' = R_2/y]$

$= 3y(x+y) \begin{vmatrix} 0 & 3 & 0 \\ 1 & -2 & 1 \\ x+y & x+2y & x \end{vmatrix}$ $[R_1' = R_1 - R_2]$

$= 3y(x+y)[0 + 3(x+y) - x + 0]$ [expansion by first row]

$= 3y(x+y)(3y) =$

$9y^2(x+y)$

Question: 14

Solution:

$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$ $[C_1' = C_1 + C_2 + C_3]$

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$$= (x + y + z)\begin{vmatrix} 1 & -x + y & -x + z \\ 1 & 3y & z - y \\ 1 & y - z & 3z \end{vmatrix} [C_1' = C_1/(x + y + z)]$$

$$= (x + y + z)\begin{vmatrix} 1 & 1 & 1 \\ -x + y & 3y & y - z \\ -x + z & z - y & 3z \end{vmatrix} \text{[transforming row and column]}$$

$$= (x + y + z)\begin{vmatrix} 0 & 0 & 1 \\ -x - 2y & 2y + z & y - z \\ -x + y & -y - 2z & x \end{vmatrix} [C_1' = C_1 - C_2 \ \& \ C_2' = C_2 - C_3]$$

= (x + y + z)[0 + 0 + (- x - 2y)(- y - 2z) - (- x + y)(2y + z)] [expansion by first row]

= $(x + y + z)(xy + 2y^2 + 2xz + 4yz + 2xy - 2y^2 + xz - yz)$

= (x + y + z)(3xy + 3yz + 3xz)

= 3(x + y + z)(xy + yz+zx)

Question: 15

Solution:

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} [C_1' = C_1/x, C_2' = C_2/y \ \& \ C_3' = C_3/z]$$

$$= xyz\begin{vmatrix} 0 & 0 & 1 \\ x - y & y - z & z \\ x^2 - y^2 & y^2 - z^2 & z^2 \end{vmatrix} [C_1' = C_1 - C_2 \ \& \ C_2' = C_2 - C_3]$$

$$= xyz\begin{vmatrix} 0 & 0 & 1 \\ x - y & y - z & z \\ (x + y)(x - y) & (y + z)(y - z) & z^2 \end{vmatrix}$$

$$= xyz(x - y)(y - z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x + y & y + z & z^2 \end{vmatrix} [C_1' = C_1/(x - y) \ \& \ C_2' = C_2/(y - z)]$$

= xyz(x - y)(y - z)(0 + 0 + y + z - x - y) [expansion by first row]

= xyz(x - y)(y - z)(z –x

Question: 16

Solution:

$$\begin{vmatrix} b + c & a - b & a \\ c + a & b - c & b \\ a + b & c - a & c \end{vmatrix}$$

$$= \begin{vmatrix} 2(a + b + c) & 0 & a + b + c \\ c + a & b - c & b \\ a + b & c - a & c \end{vmatrix} [R_1' = R_1 + R_2 + R_3]$$

$$= (a + b + c)\begin{vmatrix} 2 & 0 & 1 \\ c + a & b - c & b \\ a + b & c - a & c \end{vmatrix} [R_1' = R_1/(a + b + c)]$$

= (a + b + c)[2(b - c)c - b(c - a) + (c + a)(c - a) - (a + b)(b - c)] [expansion by first row]

= $(a + b + c)(2bc - 2c^2 - bc + ab + c^2 - a^2 - ab - b^2 + ac + bc$

= (a + b + c)(ab + bc + ac - a^2 - b^2 - c^2)

= 3abc - a^3 - b^3 -c^3

Question: 17

Solution:

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 2(b+c) & 2(a+c) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix} [R_1' = R_1 + R_2 + R_3]$$

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} [R_1' = R_1/2]$$

$$= 2\begin{vmatrix} c & 0 & a \\ b-c & a & -a \\ c & c & a+b \end{vmatrix} [R_1' = R_1 - R_2 \ \& \ R_2' = R_2 - R_3]$$

= 2[c{a(a + b) - (- ac)} + 0 + a{c(b - c) - ac}] [expansion by first row]

= 2(a^2c + abc + ac^2 + abc - ac^2 - a^2c)

= 4abc

Question: 18

Solution:

$$\begin{vmatrix} a & a+2b & a+2b+3c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix}$$

$$= \left(\frac{1}{3}\right)\begin{vmatrix} 3a & 3a+6b & 3a+6b+9c \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} [R_1' = 3R_1]$$

$$= \left(\frac{1}{3}\right)\begin{vmatrix} 0 & -a & -2a-b \\ 3a & 4a+6b & 5a+7b+9c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} [R_1' = R_1 - R_2]$$

$$= \left(\frac{1}{6}\right)\begin{vmatrix} 0 & -a & -2a-b \\ 6a & 8a+12b & 10a+14b+18c \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} [R_2' = 2R_2]$$

$$= \left(\frac{1}{6}\right)\begin{vmatrix} 0 & -a & -2a-b \\ 0 & -a & -a-b \\ 6a & 9a+12b & 11a+15b+18c \end{vmatrix} [R_2' = R_2 - R_3]$$

= (1/6)[0 + 0 + 6a{a(a + b) - a(2a + b) [expansion by first column]

= - a^3

Question: 19

Solution:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

$$= \begin{vmatrix} a+b & a+b & -(a+b) \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} [R_1' = R_1 + R_2]$$

$$= (a+b)\begin{vmatrix} 1 & 1 & -1 \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} [R_1' = R_1/(a+b)]$$

$$= (a+b)\begin{vmatrix} 1 & 1 & -1 \\ -c-b & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix} [R_2' = R_2 + R_3]$$

$$= (a+b)\begin{vmatrix} 1 & 1 & -1 \\ -(b+c) & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix}$$

$$= (a+b)(b+c)\begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} [R_2' = R_1/(b+c)]$$

$$= (a+b)(b+c)\begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix} [R_1' = R_1 + R_2]$$

$= (a + b)(b + c)\{0 + 2(- b + a + b + c) + 0\}$ [expansion by first row]

$= 2(a + b)(b + c)(c + a)$

Question: 20

Solution:

$$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$$

$$= \left(\frac{1}{xy}\right)\begin{vmatrix} ax & bx & ax^2+bxy \\ by & cy & bxy+cy^2 \\ ax+by & bx+cy & 0 \end{vmatrix} [R_1' = xR_1 \ \& \ R_2' = yR_2]$$

$$= \left(\frac{1}{xy}\right)\begin{vmatrix} 0 & 0 & ax^2+2bxy+cy^2 \\ by & cy & bxy+cy^2 \\ ax+by & bx+cy & 0 \end{vmatrix} [R_1' = R_1 + R_2 - R_3]$$

$= (1/xy)[0 + 0 + (ax^2 + 2bxy + cy^2)\{by(bx + cy) - cy(ax + by)\}]$[expansion by first row] .

$= (1/xy)(ax^2 + 2bxy + cy^2)(b^2xy + bcy^2 - acxy - bcy^2)$

$= (b^2 - ac)(ax^2 + 2bxy+cy^2)$

Question: 21

Solution:

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ a^2+2a+1 & b^2+2b+1 & c^2+2c+1 \\ a^2-2a+1 & b^2-2b+1 & c^2-2c+1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ 4a & 4b & 4c \\ a^2-2a+1 & b^2-2b+1 & c^2-2c+1 \end{vmatrix} [R_2' = R_2 - R_3]$$

$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ a^2-2a+1 & b^2-2b+1 & c^2-2c+1 \end{vmatrix}$ [R₂' = R₂/4]

$= 4\begin{vmatrix} a^2 & a & a^2-2a+1 \\ b^2 & b & b^2-2b+1 \\ c^2 & c & c^2-2c+1 \end{vmatrix}$ [transforming row and column]

$= 4\begin{vmatrix} a^2-b^2 & a-b & (a^2-b^2)-2(a-b) \\ b^2-c^2 & b-c & (b^2-c^2)-2(b-c) \\ c^2 & c & c^2-2c+1 \end{vmatrix}$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 - R_3$]

$= 4\begin{vmatrix} (a-b)(a+b) & a-b & (a-b)(a+b-2) \\ (b-c)(b+c) & b-c & (b-c)(b+c-2) \\ c^2 & c & c^2-2c+1 \end{vmatrix}$

$= 4(a-b)(b-c)\begin{vmatrix} a+b & 1 & a+b-2 \\ b+c & 1 & b+c-2 \\ c^2 & c & c^2-2c+1 \end{vmatrix}$ [$R_1' = R_1/(a-b)$ & $R_2' = R_2/(b-c)$]

$= 4(a-b)(b-c)\begin{vmatrix} a-c & 0 & a-c \\ b+c & 1 & b+c-2 \\ c^2 & c & c^2-2c+1 \end{vmatrix}$ [$R_1' = R_1 - R_2$]

$= 4(a-b)(b-c)(a-c)\begin{vmatrix} 1 & 0 & 1 \\ b+c & 1 & b+c-2 \\ c^2 & c & c^2-2c+1 \end{vmatrix}$ [$R_1' = R_1/(a-c)$]

$= 4(a - b)(b - c)(a - c)(c^2 - 2c + 1 - bc - c^2 + 2c + 0 + bc + c^2 - c^2)$ [expansion by first row]

$= 4(a - b)(b - c)(c - a)$

Question: 22

Solution:

$\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$

$= \begin{vmatrix} x^2-4x+4 & x^2-2x+1 & x^2 \\ x^2-2x+1 & x^2 & x^2+2x+1 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$

$= \begin{vmatrix} -2x+3 & -2x+1 & -2x-1 \\ -2x+1 & -2x-1 & -2x-3 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 -$

R_3] $= \begin{vmatrix} 2 & 2 & 2 \\ -2x+1 & -2x-1 & -2x-3 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$ [$R_1' = R_1 - R_2$]

$= 2\begin{vmatrix} 1 & 1 & 1 \\ -2x+1 & -2x-1 & -2x-3 \\ x^2 & x^2+2x+1 & x^2+4x+4 \end{vmatrix}$ [R₁' = R₁/2]

$= 2\begin{vmatrix} 1 & -2x+1 & x^2 \\ 1 & -2x-1 & x^2+2x+1 \\ 1 & -2x-3 & x^2+4x+4 \end{vmatrix}$ [transforming row and column]

$= 2\begin{vmatrix} 0 & 2 & -2x-1 \\ 0 & 2 & -2x-3 \\ 1 & -2x-3 & x^2+4x+4 \end{vmatrix}$ [$R_1' = R_1 - R_2$ & $R_2' = R_2 -$

R_3] $= 2\begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & -2x-3 \\ 1 & -2x-3 & x^2+4x+4 \end{vmatrix}$ [$R_1' = R_1 - R_2$]

$= 2\{0 + 0 + 2(0 - 2)\}$ [expansion by first row]

. = - 8

Question: 23

Solution:

$$\begin{vmatrix} (m+n)^2 & l^2 & mn \\ (n+l)^2 & m^2 & ln \\ (l+m)^2 & n^2 & lm \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} m^2+2mn+n^2 & l^2 & 2mn \\ n^2+2nl+l^2 & m^2 & 2ln \\ l^2+2lm+m^2 & n^2 & 2lm \end{vmatrix} [C_3' = 2C_3]$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} m^2+n^2 & l^2 & 2mn \\ n^2+l^2 & m^2 & 2ln \\ l^2+m^2 & n^2 & 2lm \end{vmatrix} [C_1' = C_1 - C_3]$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} l^2+m^2+n^2 & l^2 & 2mn \\ l^2+m^2+n^2 & m^2 & 2ln \\ l^2+m^2+n^2 & n^2 & 2lm \end{vmatrix} [C_1' = C_1 + C_2]$$

$$= \left(\frac{1}{2}\right)(l^2+m^2+n^2)\begin{vmatrix} 1 & l^2 & 2mn \\ 1 & m^2 & 2ln \\ 1 & n^2 & 2lm \end{vmatrix} [C_1' = C_1/(l^2+m^2+n^2)]$$

$$= \left(\frac{1}{2}\right)(l^2+m^2+n^2)\begin{vmatrix} 1 & 1 & 1 \\ l^2 & m^2 & n^2 \\ 2mn & 2ln & 2lm \end{vmatrix} \text{[transforming row and column]}$$

$$= \left(\frac{1}{2}\right)(l^2+m^2+n^2)\begin{vmatrix} 0 & 0 & 1 \\ l^2-m^2 & m^2-n^2 & n^2 \\ -2n(l-m) & -2l(m-n) & 2lm \end{vmatrix} [C_1' = C_1 - C_2 \ \& \ C_2' = C_2 - C_3]$$

$$= (l^2+m^2+n^2)(l-m)(m-n)\begin{vmatrix} 0 & 0 & 1 \\ l+m & m+n & n^2 \\ -n & -l & lm \end{vmatrix} [C_1' = C_1/(l-m) \ \& \ R_2' = C_2/(l-m)]$$

= $(l^2 + m^2 + n^2)(l - m)(m - n)\{0 + 0 - l(l + m) + n(m + n)\}$ [expansion by first row]

= $(l^2 + m^2 + n^2)(l - m)(m - n)\{0 + 0 - l(l + m) + n(m + n)\}$

= $(l^2 + m^2 + n^2)(l - m)(m - n)(- l^2 - ml + mn + n^2)$

= $(l^2 + m^2 + n^2)(l - m)(m - n)\{(n^2 - l^2) + m(n - l)\}$

= $(l^2 + m^2 + n^2)(l - m)(m - n)(n - l)(l + m + n)$

Question: 24

Solution:

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} b^2+2bc+c^2 & a^2 & 2bc \\ c^2+2ac+a^2 & b^2 & 2ca \\ a^2+2ab+b^2 & c^2 & 2ab \end{vmatrix} [C_3' = 2C_3]$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} b^2+c^2 & a^2 & 2bc \\ c^2+a^2 & b^2 & 2ca \\ a^2+b^2 & c^2 & 2ab \end{vmatrix} [C_1' = C_1 - C_3]$$

$$= \left(\frac{1}{2}\right)\begin{vmatrix} a^2+b^2+c^2 & a^2 & 2bc \\ a^2+b^2+c^2 & b^2 & 2ca \\ a^2+b^2+c^2 & c^2 & 2ab \end{vmatrix} [C_1' = C_1 + C_2]$$

$= \left(\frac{1}{2}\right)(a^2 + b^2 + c^2)\begin{vmatrix} 1 & a^2 & 2bc \\ 1 & b^2 & 2ca \\ 1 & c^2 & 2ab \end{vmatrix}$ [$C_1' = C_1/(a^2 + b^2 + c^2)$]

$= \left(\frac{1}{2}\right)(a^2 + b^2 + c^2)\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ 2bc & 2ca & 2ab \end{vmatrix}$ [transforming row and column]

$= \left(\frac{1}{2}\right)(a^2 + b^2 + c^2)\begin{vmatrix} 0 & 0 & 1 \\ a^2 - b^2 & b^2 - c^2 & c^2 \\ -2c(a-b) & -2a(b-c) & 2ab \end{vmatrix}$ [$C_1' = C_1 - C_2$ & $C_2' = C_2 - C_3$]

$= (a^2 + b^2 + c^2)(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ a + b & b + c & c^2 \\ -c & -a & ab \end{vmatrix}$ [$C_1' = C_1/(a - b)$ & $C_2' = C_2/(b - c)$]

$= (a^2 + b^2 + c^2)(a - b)(b - c)\{0 + 0 - a(a + b) + c(b + c)\}$ [expansion by first row]

$= (a^2 + b^2 + c^2)(a - b)(b - c)\{0 + 0 - a(a + b) + c(b + c)\}$

$= (a^2 + b^2 + c^2)(a - b)(b - c)(- a^2 - ba + bc + c^2)$

$= (a^2 + b^2 + c^2)(a - b)(b - c)\{(c^2 - a^2) + b(c - a)\}$

$\mathbf{= (a^2 + b^2 + c^2)(a - b)(b - c)(c - a)(a + b + c)}$

Question: 25

Solution:

$\begin{vmatrix} b^2 + c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$

$= \begin{vmatrix} 2(b^2 + c^2) & 2(c^2 + a^2) & 2(a^2 + b^2) \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$ [$R_1' = R_1 + R_2 + R_3$]

$= 2\begin{vmatrix} (b^2 + c^2) & (c^2 + a^2) & (a^2 + b^2) \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$ [$R_1' = R_1/2$]

$= 2\begin{vmatrix} c^2 & 0 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix}$ [$R_1' = R_1 - R_2$]

$= 2[c^2\{(c^2 + a^2)(a^2 + b^2) - b^2c^2\} + 0 + a^2\{b^2c^2 - c^2(c^2 + a^2)\}]$ [expansion by first row]

$= 2[c^2(c^2a^2 + a^4 + b^2c^2 + a^2b^2 - b^2c^2) + a^2(b^2c^2 - c^4 - a^2c^2)]$

$= 2[a^2c^4 + a^4c^2 + a^2b^2c^2 + a^2b^2c^2 - a^2c^4 - a^4c^2]$

$\mathbf{= 4a^2b^2c^2}$

Question: 26

Solution:

Operating $R_1 \rightarrow R_1 + bR$ � $R_2 \rightarrow R_2 - aR_3$

$\begin{vmatrix} 1 + a^2 - b^2 + 2b^2 & 2ab - 2ab & -2b + b - a^2b - b^3 \\ 2ab - 2ab & 1 - a^2 + b^2 + 2a^2 & 2a - a + a^3 + ab^2 \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix}$

$= \begin{vmatrix} 1 + a^2 + b^2 & 0 & -b - a^2b - b^3 \\ 0 & 1 + a^2 + b^2 & a + a^3 + ab^2 \\ 2b & -2a & 1 - a^2 + b^2 \end{vmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -b(1+a^2+b^2) \\ 0 & 1+a^2+b^2 & a(1+a^2+b^2) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Taking $(1+a^2+b^2)$ from R_1 and R_2

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Operating $R_3 \rightarrow R_3 - 2bR_1 + 2aR_2$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$

Taking $(1+a^2+b^2)$ from R_3

$$(1+a^2+b^2)^3 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding with respect to C_1

$= (1+a^2+b^2)^3\ 1\times[1-0]$

$= (1+a^2+b^2)^3$

Hence proved

Question: 27

Solution:

Operating $C_1 \rightarrow aC_1$

$$\frac{1}{a} \begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & a+b & c \end{vmatrix}$$

Operating $C_1 \rightarrow C_1 + bC_2 + cC_3$

$$= \frac{1}{a} \begin{vmatrix} a^2+b^2-bc+c^2+bc & b-c & c+b \\ a^2+ac+b^2+c^2-ac & b & c-a \\ a^2-ab+ab+b^2+c^2 & a+b & c \end{vmatrix}$$

$$= \frac{1}{a} \begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & a+b & c \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from C_1

$$= \frac{1}{a}(a^2+b^2+c^2) \begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & a+b & c \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - R_3$

$$= \frac{1}{a}(a^2+b^2+c^2) \begin{vmatrix} 0 & -c-a & b \\ 0 & -a & -a \\ 1 & a+b & c \end{vmatrix}$$

Operating $C_2 \rightarrow C_2 - C_3$

$$= \frac{1}{a}(a^2+b^2+c^2) \begin{vmatrix} 0 & -(a+b+c) & b \\ 0 & 0 & -a \\ 1 & (a+b+c) & c \end{vmatrix}$$

Taking $(a+b+c)$ common from C_2

$$= \frac{1}{a}(a^2+b^2+c^2)(a+b+c)\begin{vmatrix} 0 & -1 & b \\ 0 & 0 & -a \\ 1 & 1 & c \end{vmatrix}$$

Expanding with respect to C_1

$$= \frac{1}{a}(a^2+b^2+c^2)(a+b+c) \times 1 \times \left(0 - (-a)\right)$$

$$= \frac{1}{a}(a^2+b^2+c^2)(a+b+c)\ (a)$$

$= (a^2+b^2+c^2)(a+b+c)$

Question: 28

Solution:

Expanding with R1

$=_b{}^2c^2(a^2c+abc-abc-a^2b)-bc(a^3c^2+a^2bc^2-a^2b^2c-a^3b^2)+(b+c)(a^3bc^2-$

$=_a a^3b^2c)\ {}^2b^3c^2-a^2b^3c^2-a^3bc^2-a^2b^3c^2+a^2b^3c^2+a^3b^3c+a^3b^2c^2-$

$a^3b^3c+a^3bc^3-a^3b^2c^2$

$=0$

Question: 29

Solution:

$$= \begin{vmatrix} b^2+c^2+2bc & ab & ac \\ ab & a^2+c^2+2ac & bc \\ ac & bc & a^2+b^2+2ab \end{vmatrix}$$

Operating $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$

$$= \frac{1}{abc}\begin{vmatrix} a(b^2+c^2+2bc) & a^2b & a^2c \\ ab^2 & b(a^2+c^2+2ac) & b^2c \\ ac^2 & bc^2 & c(a^2+b^2+2ab) \end{vmatrix}$$

Taking a, b, c common from C_1, C_2, C_3 respectively

$$= \frac{abc}{abc}\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (a+c)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (a+c)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (a+c+b)(a+c-b) & b^2 \\ (c-a-b)(c+a+b) & (c-a-b)(c+a+b) & (a+b)^2 \end{vmatrix}$$

Taking (a+b+c) common from R_1, R_2

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Operating $R_3 \to R_3 - R_1 - R_2$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ -2b & -2a & a^2+b^2+2ab-a^2-b^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & a+c-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

Operating $C_1 \rightarrow aC_1, C_2 \rightarrow bC_2$

$$\frac{(a+b+c)^2}{ab} \begin{vmatrix} a(b+c-a) & 0 & a^2 \\ 0 & b(a+c-b) & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$

Operating $C_1 \rightarrow C_1 + C_3, C_2 \rightarrow C_2 + C_3$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(a+c) & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

Taking a, b, 2ab from R_1, R_2, R_3

$$= \frac{(a+b+c)^2 a.b.2ab}{ab} \begin{vmatrix} b+c & a & a \\ b & a+c & b \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding with R_3

$$= 2ab(a+b+c)^2 \times 1 \times (ab+ac+bc+c^2-ab)$$

$$= 2ab(a+b+c)^2 (c(a+b+c))$$

$$= 2abc(a+b+c)^3$$

Question: 30

Solution:

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

Taking (b-a) common from C_1, C_3

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

Operating $R_2 \rightarrow R_2 - R_1 + R_3$

$$= \begin{vmatrix} b & b-c-b+c & c \\ a & a-b-a+b & b \\ c & c-a-c+a & a \end{vmatrix}$$

$$= (b-a)^2 \begin{vmatrix} b & 0 & c \\ a & 0 & b \\ c & 0 & a \end{vmatrix}$$

[Properties of determinants say that if 1 row or column has only 0 as its elements, the value of thedeterminant is 0]

$= 0$

Hence Proved

Question: 31

Solution:

Taking a, b, c from C_1, C_2, C_3

$$= abc\begin{vmatrix} -b^2-c^2+a^2 & 2b^2 & 2c^2 \\ 2a^2 & b^2-c^2-a^2 & 2c^2 \\ 2a^2 & 2b^2 & -a^2-b^2+c^2) \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - R_3$

$$= abc\begin{vmatrix} -b^2-c^2-a^2 & 0 & a^2+b^2+c^2 \\ 0 & -(a^2+b^2+c^2) & a^2+b^2+c^2 \\ 2a^2 & 2b^2 & -a^2-b^2+c^2 \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from R_1, R_2

$$= abc(a^2+b^2+c^2)^2\begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2a^2 & 2b^2 & -a^2-b^2+c^2 \end{vmatrix}$$

Operating $R_3 \rightarrow R_3 + R_1 + R_2$

$$= abc(a^2+b^2+c^2)^2\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2a^2 & 2b^2 & a^2+b^2+c^2 \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$ common from C_3 �

$$= abc(a^2+b^2+c^2)^3\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2a^2 & 2b^2 & 1 \end{vmatrix}$$

Expanding with C_3

$= abc(a^2+b^2+c^2)^3 \times 1 \times (1-0)$

$= abc$

$(a^2+b^2+c^2)^3$

Hence proved

Question: 32

Solution:

Given that α, β, γ are in an AP, which means $2\beta=\alpha+\gamma$

Operating $R_3 \rightarrow R_3 - 2R_2 + R_1$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1-2x+4+x-3 & x-2-2x+6+x-4 & x-\gamma-2x+2\beta+x-\alpha \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ 0 & 0 & -\gamma+2\beta-\alpha \end{vmatrix}$$ [we know that $2\beta=\alpha+\gamma$]

Operating $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R$ �R_3

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ 0 & 0 & -\gamma+\alpha+\gamma-\alpha \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ 0 & 0 & 0 \end{vmatrix}$$

[By the properties of determinants, we know that if all the elements of a row or column is 0, thenthe value of the determinant is also 0]

$=0$

Hence proved

Question: 33

Solution:

Operating $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} (a+1)(a+2)-(a+2)(a+3) & a+2-a-3 & 0 \\ (a+2)(a+3)-(a+3)(a+4) & a+3-a-4 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (a+2)(a+1-a-3) & -1 & 0 \\ (a+3)(a+2-a-4) & -1 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -2(a+2) & -1 & 0 \\ -2(a+3) & -1 & 0 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

Expanding with C_3

= (2(a+2) – 2(a+3))

= (2a+4-2a-6)

= -2

Question: 34

Solution:

By properties of determinants, we can split the given determinant into 2 parts

$$\to 0 = \begin{vmatrix} x & x^3 & x^4 \\ y & y^3 & y^4 \\ z & z^3 & z^4 \end{vmatrix} - \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$$

Taking x, y, z common from R_1, R_2, R_3 respectively

$$\to 0 = xyz\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} - \begin{vmatrix} x & x^3 & 1 \\ y & y^3 & 1 \\ z & z^3 & 1 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$\to 0 = xyz\begin{vmatrix} 0 & x^2-z^2 & x^3-z^3 \\ 0 & y^2-z^2 & y^3-z^3 \\ 1 & z^2 & z^3 \end{vmatrix} - \begin{vmatrix} x-z & x^3-z^3 & 0 \\ y-z & y^3-z^3 & 0 \\ z & z^3 & 1 \end{vmatrix}$$

$$\to \begin{vmatrix} x-z & (x-z)(x^2+xz+z^2) & 0 \\ y-z & (y-z)(y^2+yx+z^2) & 0 \\ z & z^3 & 1 \end{vmatrix} = xyz\begin{vmatrix} 0 & (x-z)(x+z) & (x-z)(x^2+xz+z^2) \\ 0 & (y-z)(y+z) & (y-z)(y^2+yz+z^2) \\ 1 & z^2 & z^3 \end{vmatrix}$$

Taking (x-z) and (y-z) common from R_1, R_2

$$\to (x-z)(y-z)\begin{vmatrix} 1 & (x^2+xz+z^2) & 0 \\ 1 & (y^2+yz+z^2) & 0 \\ z & z^3 & 1 \end{vmatrix} = (x-z)(y-z)\begin{vmatrix} 0 & x+z & (x^2+xz+z^2) \\ 0 & y+z & (y^2+yz+z^2) \\ 1 & z^2 & z^3 \end{vmatrix}$$

Expanding with R_3

$\to y^2+yz+z^2-x^2-xz-z^2 = xyz(xy^2+xyz+xz^2+zy^2+yz^2+z^3-x^2y-xyz-yz^2-x^2z-xz^2-z^3)$

$\to (y-x)(y+x) + z(y-x) = xyz(xy^2+zy^2 - x^2y - x^2z)$

$\to (y-x)(x+y+z) = xyz(xy(y-x)+z(y^2-x^2))$

$\to (y-x)(x+y+z) = xyz(xy(y-x)+z(x+y)(y-x))$

$\rightarrow$(y-x)(x+y+z) = xyz(xy(y-x)+(xz+yz)(y-x))

$\rightarrow$(y-x)(x+y+z)= xyz(y-x)(xy+xz+yz)

$\rightarrow$x+y+z =

xyz(xy+xz+yz)Hence

Proved

Question: 35

Solution:

Operating $R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$

$$= \begin{vmatrix} 0 & a^2+bc-b^2-ac & a^3-b^3 \\ 0 & b^2+ca-c^2-ab & b^3-c^3 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & (a-b)(a+b)-c(a-b) & (a-b)(a^2+ab+b^2) \\ 0 & (b-c)(b+c)-a(b-c) & (b-c)(b^2+bc+c^2) \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Taking (a-b), (b-c) common from R_1, R_2 respectively

$$= (a-b)(b-c)\begin{vmatrix} 0 & a+b-c & a^2+ab+b^2 \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 - R_2$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 2a-2c & a^2+ab-bc-c^2 \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 2(a-c) & (a+c)(a-c)+b(a-c) \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Taking (a-c) common from R_1

$$= (a-c)(a-b)(b-c)\begin{vmatrix} 0 & 2 & a+b+c \\ 0 & b+c-a & b^2+bc+c^2 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Expanding with C_1

= (a-c)(a-b)(b-c)×($2b^2+2bc+2c^2-ab-b^2-bc-ac-bc-c^2+a^2+ab+ac$)

=-(c-a)(b-c)(a-b)($a^2+b^2+c^2$)

Hence Proved

Question: 36

Operating $R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$

$$\rightarrow \begin{vmatrix} 0 & a-b & bc-ac \\ 0 & b-c & ac-ab \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} 0 & a-b & -c(a-b) \\ 0 & b-c & -a(b-c) \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

Taking (a-b) and (b-c) from R_1, R_2

$$\rightarrow (a-b)(b-c)\begin{vmatrix} 0 & 1 & -c \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 1 & (a+b) \\ 0 & 1 & (b+c) \\ 1 & c & c^2 \end{vmatrix}$$

Method 1:

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For the two determinants to be equal, their difference must be 0.

$$= \begin{vmatrix} 0 & 1 & -c \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix} - \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0-0 & 1-1 & -(a+b+c) \\ 0-0 & 1-1 & -(a+b+c) \\ 1-1 & c-c & ab-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -(a+b+c) \\ 0 & 0 & -(a+b+c) \\ 0 & 0 & ab-c^2 \end{vmatrix}$$

Since 2 columns have only 0 as their elements, by properties of determinants

=0

Method 2:

Expanding both with

C_1LHS

=(a-b)(b-c)(-a+c)

RHS

=(a-b)(b-c)(b+c-a-b)

=(a-b)(b-c)(-a+c)

∴ LHS = RHS

Question: 37

Operating $R1 \rightarrow R1 - R_3$, $R_2 \rightarrow R_2 - R3$

$$\begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & bc-ab & b+c-a-b \\ 0 & ac-ab & c+a-a-b \\ 1 & ab & a+b \end{vmatrix}$$

$$\rightarrow \begin{vmatrix} 0 & a-c & (a-c)(a+c) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & -b(a-c) & -(a-c) \\ 0 & -a(b-c) & -(b-c) \\ 1 & ab & a+b \end{vmatrix}$$

Taking (a-c) and (b-c) common from R_1, R_2

$$\rightarrow (a-c)(b-c) \begin{vmatrix} 0 & 1 & a+c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-c)(b-c) \begin{vmatrix} 0 & -b & -1 \\ 0 & -a & -1 \\ 1 & ab & a+b \end{vmatrix}$$

Method 1:

If the determinants are equal, their difference must also be

equal.(a-c) and (b-c) get cancelled.

$$= \begin{vmatrix} 0 & 1 & a+c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 0 & -b & -1 \\ 0 & -a & -1 \\ 1 & ab & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 0-0 & 1+b & a+c+1 \\ 0-0 & 1+a & b+c+1 \\ 1-1 & c-ab & c^2+a+b \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1+b & a+c+1 \\ 0 & 1+a & b+c+1 \\ 0 & c-ab & c^2+a+b \end{vmatrix}$$

Since all elements of C_1 are 0, by properties of determinants,

=0

∴ The 2 determinants are equal.

Method 2:

Expanding with C_1

→(a-c)(b-c)(b+c-a-c) = (a-c)(b-c)(b-a)

→(a-c)(b-c)(b-a)=(a-c)(b-c)(b-a)

∴ RHS and LHS are equal

Question: 38 i

Solution:

Operating $R_1 \to R_1 - R_2$

$$0 = \begin{vmatrix} x-2 & -6+3x & -1-x+3 \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix}$$

$$0 = \begin{vmatrix} x-2 & 3(x-2) & -(x-2) \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix}$$

Taking (x-2) common from R_1

$$0 = (x-2)\begin{vmatrix} 1 & 1 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & 2+x \end{vmatrix}$$

Here, we can see that x-2 is a factor of the determinant.

We can say that when x-2 is put in the equation, we get

0.x-2=0

→x=2

Question: 39

Solution:

Operating $R1 \to R1 - R_2$, $R_2 \to R2 - R_3$

$$\begin{vmatrix} 0 & x-b & x^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

$$0 = \begin{vmatrix} 0 & x-c & (x-b)^3+3xb(x-b) \\ 0 & b-c & (b-c)^3+3bc(b-c) \\ 1 & c & c^3 \end{vmatrix}$$

$$0 = (x-c)(b-c)\begin{vmatrix} 0 & 1 & (x-b)^2+3xb \\ 0 & 1 & (b-c)^2+3bc \\ 1 & c & c^3 \end{vmatrix}$$

Expanding with C_1

$0=(x-c)(b-c)(b^2-2bc+c^2+3bc-x^2+2xb-b^2-$

$3xb)0= (x-c)(b-c)(bc+c^2-x^2-xb)$

0=(x-c)(b-c)(-b(-c+x)-(c-x)(-c-x))

$0=(x-c)^2(b-c)(-b-c-x)$

Either x-c=0 or b-c=0 or (-b-c-x)=0

∴.x=c or b=c or x=-

(b+c)If b=c, x=b

∴. x=c or x=b or x=-(b+c)

Question: 40

Solution:

Operating $C1 \to C1 + C_2 + C_3$

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix} = 0$$

Taking (x+a+b+c) common from C_1

$$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix} = 0$$

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$(x+a+b+c)\begin{vmatrix} 0 & 0 & -x \\ 0 & x & -x \\ 1 & b & x+c \end{vmatrix} = 0$$

Expanding with C_1

0=(x+a+b+c)(0+x^2

)

0=x^2(x+a+b+c)

Either x^2=0 or (x+a+b+c)=0

∴. x=0 or x=-(a+b+c)

Question: 41

Solution:

Operating $C1 \to C1 + C_2 + C_3$

$$0 = \begin{vmatrix} 3x-8+3+3 & 3 & 3 \\ 3+3x-8+3 & 3x-8 & 3 \\ 3+3+3x-8 & 3 & 3x-8 \end{vmatrix}$$

$$0 = \begin{vmatrix} 3x-2 & 3 & 3 \\ 3x-2 & 3x-8 & 3 \\ 3x-2 & 3 & 3x-8 \end{vmatrix}$$

Taking (3x-2) common from C_1

$$0 = (3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$$0 = (3x-2)\begin{vmatrix} 0 & 0 & -(3x-11) \\ 0 & 3x-11 & -3x+11 \\ 1 & 3 & 3x-8 \end{vmatrix}$$

Expanding with C_1

0= (3x-2)(0+(3x-

11)^2)

$0=(3x-2)(3x-11)^2$

Either 3x-2=0 or 3x-11=0

$\therefore x=\frac{2}{3}$ or $x=\frac{11}{3}$

Question: 42

Solution:

Operating $C_1 \to C_1+C_2+C_3$

$$0=\begin{vmatrix} x+9 & 3 & 5 \\ x+9 & x+2 & 5 \\ x+9 & 3 & x+4 \end{vmatrix}$$

Taking (x+9) common from C_1

$$0=(x+9)\begin{vmatrix} 1 & 3 & 5 \\ 1 & x+2 & 5 \\ 1 & 3 & x+4 \end{vmatrix}$$

Operating $R_1 \to R_1-R_3$, $R_2 \to R_2-R_3$

$$0=(x+9)\begin{vmatrix} 0 & 0 & 1-x \\ 0 & x-1 & 1-x \\ 1 & 3 & x+4 \end{vmatrix}$$

$0=(x+9)(0-x+x^2+1-x)$

$0=(x+9)(x^2-2x+1)$

$0=(x+9)(x-1)^2$

$\therefore$ Either x+9=0 or x-

1=0x=-9, x=1

Question: 43

Solution:

Operating $R1 \to R1+R_2+R_3$

$$0=\begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

Taking (x+9) common from R_1

$$0=(x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

Operating $C_1 \to C_1-C_3$, $C_2 \to C_2-C_3$

$$0=(x+9)\begin{vmatrix} 0 & 0 & 1 \\ 0 & x-2 & 2 \\ 7-x & 6-x & x \end{vmatrix}$$

Expanding with R_1

$0=(x+9)(0-(x-2)(7-$

$x))$

$0=(x+9)(7-x)(2-x)$

Either x+9=0 or 7-x=0 or 2-x=0

$\therefore$ x=-9 or x=7 or x=2

Question: 44

Solution:

Expanding with R1

$0=x(-3x^2-6x-2x^2+6x)+6(2x+4+3x-9)-1(4x-9x)$

$0=x(-5x^2)+6(5x-5)-1(-5x)$

$0=-5x^3+30x-30+5x$

$0=-5x^3+35x-30$

$x^3-7x+6=0$

$x^3-x-6x+6=0$

$x(x^2-1)-6(x-1)=0$

x(x-1)(x+1)-6(x-

1)=0

$(x-1)(x^2+x-6)=0$

$(x-1)(x^2+3x-2x-$

6)=0(x-1)(x(x+3)-

2(x+3)(=0

(x-1)(x+3)(x-2)=0

Either x-1=0 or x+3=0 or x-2=0

∴ x=1 or x=-3 or x=2

Question: 45

Solution:

Operating $C_1 \rightarrow aC_1$

$$=\frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix}$$

Operating $C_1 \rightarrow C_1+bC_2+cC_3$

$$=\frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix}$$

Taking $(a^2+b^2+c^2)$

$$=\frac{a^2+b^2+c^2}{a}\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

Operating $C_2 \rightarrow C_2-bC_1$, $C_3 \rightarrow C_3-cC_3$

$$=\frac{a^2+b^2+c^2}{a}\begin{vmatrix} 1 & -c & b \\ 1 & 0 & -a \\ 1 & a & 0 \end{vmatrix}$$

Expanding with R_3

$$=\frac{a^2+b^2+c^2}{a}(ac-0+a^2+ab)$$

$$=\frac{a^2+b^2+c^2}{a}a(a+b+c)$$

=

$(a^2+b^2+c^2)(a+b+c)$

Hence Proved